REVIEW

by Prof. Andrey Ivanov Zahariev, PhD

Department of Mathematical Analysis, Faculty of Mathematics and Informatics, Plovdiv University "Paisii Hilendarskii"

on a PhD Thesis for the award of the educational and scientific degree PhD

in: field of higher education: 4. Natural Sciences, Mathematics and Informatics;

professional field: 4.5. Mathematics

doctoral program: Mathematical Analysis

Author: Laura Ajeti Azemi - doctoral student in an independent form of study

Title: On Coupled Fixed Points and Coupled Best Proximity Points for Cyclic, Noncyclic, and Semi-Cyclic Maps.

Scientific supervisors: Prof. Dr. Sci. Boyan Georgiev Zlatanov

Assoc. Prof Hristina Nikolova Kulina, PhD

1. General description of the submitted materials

By order No. PD-22-966 of 28.04.2025 of the Rector of Plovdiv University "Paisii Hilendarskii" (PU) I have been appointed as a member of the scientific jury for ensuring a procedure for the defense of a PhD Thesis with Title: “On Coupled Fixed Points and Coupled Best Proximity Points for Cyclic, Noncyclic, and Semi-Cyclic Maps” for the acquisition of the educational and scientific degree PhD in the field of higher education: 4. Natural Sciences, Mathematics and Informatics; professional field: 4.5. Mathematics; doctoral program: Mathematical Analysis

The author of the PhD Thesis is Laura Ajeti Azemi - a doctoral student in independent training) at the Department of Mathematical Analysis with scientific supervisors Prof. Dr. Sci. Boyan Georgiev Zlatanov and Assoc. Prof. PhD. Hristina Nikolova Kulina from Plovdiv University "Paisii Hilendarskii".

The set of materials on paper submitted by Laura Ajeti Azemi is in accordance with Art. 36 (1) of the Regulations for the Development of the Academic Staff of PU, and includes the following documents:

1. a sample application to the rector for the opening of a procedure;
2. a curriculum vitae in European format;
3. a report of the preliminary discussion in the department and a statement from the scientific supervisor regarding the readiness for preliminary discussion;
4. an abstract (in Bulgarian and a foreign language);

4.a. in Bulgarian 64 pages

4.b. in English 32 pages

4.c. in Bulgarian 32 pages

5. a declaration of originality and authenticity of the attached documents;
6. a certificate of compliance with the minimum national requirements;
7. a list of publications;
8. a PhD Thesis;
9. copies of publications on the topic of the PhD Thesis;
10. a document for the fee paid, according to the Tariff;
11. a set of documents on paper from item 1 to item 10 – 3 copies;

12. set of documents on electronic media from item 1 to item 10 – 3 pieces.

The doctoral student has attached 3 publications, two of which are journals and one in a collection of reports of the relevant conferences.

2. Brief biographical data about the doctoral student.

This information is contained in the attached autobiography in European format.

3. Relevance of the topic and appropriateness of the set goals and objectives.

A review of the SCOPUS and Web of Science databases shows that in recent years there has been a significant increase in publications dedicated to research in the field of various types of generalizations of fixed-point theorems, as well as their applications in the theory of oligopolistic markets. This reflects the globalization of the economy, in which fewer and fewer economic entities remain in the markets, which have increasingly larger shares of the relevant markets. Publications in these areas are both theoretical in mathematical journals and such applied aspects in economic journals. My opinion is that the presented research is not only relevant and will be of interest to a wide range of readers, both mathematicians and economists. I would like to note that the first publication of the doctoral student in 2022 was published in the journal Mathematics, 10(8), (2022), 1304.IF=2.258, quartile Q1 and already has 4 citations that are indexed in the WoS and/or SCOPUS databases, which is convincing evidence of both the relevance of the tasks studied in the PhD Thesis and their quality.

4. Knowledge of the problem

As is well known and in my opinion, at the present time, no researcher is able to familiarize himself with and assimilate all published results on a given scientific topic (even in a relatively "narrow" direction). The main reasons for this are: the presence of a huge number of sources, as well as externally imposed limited access to information (technically inaccessible and/or financially inaccessible). In this aspect, knowledge of a given scientific problem means that the researcher possesses a certain set of scientific information on the topic, which together provide the necessary quality, depth, scope of knowledge and are sufficient for a given specific study. In my opinion, Laura Ajeti Azemi has a thorough knowledge of both the current state and the historical scientific problems considered in her PhD Thesis. This impression of mine is based on the PhD Thesis she presented. Given the fact that the first of the publications of a doctoral student and her scientific supervisor since 2022, gives me reason to conclude that the doctoral student had the opportunity to study for more than 3 years on the topic of the PhD Thesis and to enter the topic and the methods used, which means that Laura Ajeti Azemi began her joint scientific work with one of her scientific supervisors before this year. The subsequent results were published in the period 2023 - 2025, which shows systematic and in-depth preparation, which allowed the student to present a completed PhD Thesis when enrolling as a doctoral student in an independent form of study. In conclusion, I will note that in my opinion the literature contains only sources that are directly related to the research in the PhD Thesis without parasitic citations.

5. Research methodology

The main apparatus for conducting research in the reviewed work are the methods and scientific facts from several mathematical areas: real and functional analysis; topology; variational principles; theory of oligopolistic markets, statistical methods for modeling on databases. For research on this topic, it cannot be said that exactly one method or even several well-known methods are used, but rather different knowledge, skills and techniques are applied, which the researcher manages to combine in order to solve the tasks set.

6. Characteristics and evaluation of the PhD Thesis work

The PhD Thesis work generalizes Banach's fundamental result on the existence and uniqueness of fixed points. These generalizations are in different directions, including coupled fixed points, tripled fixed points, coupled best proximity, cyclic map, non-cyclic maps and semi-cyclic ones, when the adjacent space is either uniformly convex Banach space, or a reflexive Banach space, or only a metric space. An important applied aspect is finding an estimate of the error when using series of successive approximations for pairs of optimal points (which are the natural generalization of the best proximity points for cyclic maps) for non-cyclic maps. I will note that the obtained results are illustrated not only with abstractly selected applications, but also with those that arise when studying market equilibrium in oligopolistic markets on real data. Of serious interest is the result that generalizes Ekeland's variational principle to sets generated by maps with the mixed monotone property, when the adjacent metric space is partially ordered.

7. Contributions and significance of the development for science and practice.

The PhD Thesis presented for defense is 136 pages long with the following content structure: introduction, four chapters, conclusion and bibliography of 90 titles.

The introduction consists of two parts: A preface containing a brief description of the problems studied in the PhD Thesis and Chapter 1 (entitled Preliminaries) where all the concepts and theorems necessary for the presentation of the results obtained by the doctoral student in the following chapters with numbers 2, 3 and 4 are presented.

In Chapter 2, coupled best proximity points for cyclic and semi-cyclic maps are studied. Semi-cyclic maps appear naturally when we consider a market with two participants who change their production levels, taking into account both their own results and those of the competitor. Since the classical Euclidean metric does not adequately describe the profits of the participants, the PhD student uses the equivalent summative one, which is more suitable for the study of oligopoly markets, overcoming the problem that the Banach space under consideration is not uniformly convex, but only reflexively convex. New concepts are introduced, such as coupled fixed points and coupled best proximity points for semi-cyclic mappings. The concept of a pair of optimal fixed points for noncyclic mappings is generalized to mappings of two variables, and sufficient conditions for the existence and uniqueness of ordered pairs of optimal points are found when the adjacent space is uniformly convex, and a priori and a posteriori estimates of the error are found when the modulus of convexity is of power order.

In Chapter 3, the doctoral student has generalized the fundamental Ekeland variational principle on sets generated by mappings with the mixed monotone property in partially ordered metric spaces. The obtained result was used to find sufficient conditions for the

existence and uniqueness of tripled -fixed points for ordered triples of maps with the mixed monotonic property. I will note that a number of already known results for tripled fixed points are consequences of the result obtained by the doctoral student. In my opinion, the results obtained by the PhD student in mathematical analysis described in chapters 2 and 3, both in quantity and quality, are completely sufficient for the acquisition of the educational and scientific degree PhD.

The subject of Chapter 4 is the applications of the results obtained in chapters 2 and 3 in the study of oligopolistic markets, as well as and some other unpublished results. These results are devoted to the modelling of an oligopolistic market with three participants based on real data from it. Appropriate models of response functions for participants that possess the mixed monotonic property have been constructed. A statistical analysis of the data predicted by the constructed models has been performed. The illustrative examples show that there are markets in which functions with the mixed monotonic property naturally describe the real data of the respective markets in a statistically reliable way. It is also impressive that the necessary information about the market equilibrium according to Cournot (Antoine-Augustin Cournot) for oligopolistic markets, as well as its generalization when using reaction functions instead of payoff maximization by statistical methods for estimating models using real databases, are introduced in Chapter 4, where they are used, which makes it easier for the reader. The results obtained in this chapter illustrate the possibilities for applying the obtained theoretical results in the study of market equilibrium for oligopoly markets.

The conclusion correctly describes the contributions in the presented PhD Thesis work.

8. Assessment of the publications in the PhD Thesis work

The PhD Thesis work is built on the basis of 3 publications of the doctoral student (two in journals and one in conference reports). One of the publications was published in the prestigious journal "Mathematics" with an impact factor of 2.258, quartile Q1, and the other is indexed in SCOPUS with SJR=0.152. This review shows that the doctoral student with his 105 points seriously exceeds the minimum national requirements according to the PPZRASRB of 30 points. In addition, I will note that the article in "Mathematics" already has 4 citations that are indexed in the WoS and/or SCOPUS databases, which is convincing evidence both for the relevance of the research in the published articles and for the quality of the results presented in them.

The results have been approbated at three scientific forums.

The abstract correctly systematizes the results of the PhD Thesis.

I have not found "plagiarism" in the candidate's works within the meaning of the Law on the Development of the Academic Staff in the Republic of Bulgaria.

9. Personal participation of the doctoral student

In my opinion, the doctoral candidate has a significant personal contribution to the research conducted and the results obtained.

10. Abstract

The summary correctly and correctly systematizes the results of the PhD Thesis.

11. Critical remarks and recommendations

I have no critical remarks about the clear layout of the results obtained. I recommend that the author continue his work on the topic of the PhD Thesis, to try to solve and publish the open problems set at the end of each chapter, and also to format and publish the results of chapter 4.

CONCLUSION

The PhD Thesis contains scientific and applied scientific results that represent an original contribution to science. The presented documents meet all the requirements of the Law on the Development of the Academic Staff in the Republic of Bulgaria (LAADRB), the Regulations for the Implementation of the LAADRB, the Regulations for the Development of the Academic Staff of the University of Plovdiv.

The PhD Thesis shows that the doctoral student Laura Ajeti Azemi possesses in-depth theoretical knowledge and professional skills in the scientific specialty of mathematical analysis, demonstrating qualities and skills for independent conduct of scientific research.

Due to the above, I confidently give my strictly positive assessment of the conducted research, presented by the above reviewed PhD Thesis, summary, achieved results and contributions, and I propose to the honorable scientific jury to award the educational and scientific degree PhD to Laura Ajeti Azemi in the field of higher education: 4. Natural Sciences, Mathematics and Informatics, professional field: 4.5 Mathematics, doctoral program: Mathematical Analysis.

10.05.2025 Signature: .....
Plovdiv /prof. Andrey Zahariev, PhD/