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On Coupled Fixed Points and Coupled Best Proximity Points for Cyclic, Noncyclic, and Semi-Cyclic Maps

SUMMARY

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The dissertation "On Coupled Fixed Points and Coupled Best Proximity Points for Cyclic, Noncyclic, and Semi-Cyclic Maps" contains 136 pages. It consists of a preface, introduction, 3 chapters, a conclusion, and a bibliography. The bibliography contains 90 sources. The list of author's publications on the dissertation consists of 4 items.

The public defense of the thesis will take place on 4.06.2025 at 13h in the meeting hall of the New Building of Plovdiv University "Paisii Hilendarski", Plovdiv, 236 Bulgaria Blvd.

The documents are available to those interested in the Dean's Secretary of the Faculty of Mathematics and Informatics, New building of Plovdiv University "Paisii Hilendarski", office 330, every working day from 8:30 to 17:00.

Contents

GENERAL CHARACTERISTICS OF THE THESIS

This PhD considers generalizations of the Banach fixed point theorem related to coupled fixed points, tripled fixed points, coupled best proximity points, and their applications. The generalizations concern a weakening of the assumption for the underlying space to be a uniformly convex Banach space by considering just a reflexive Banach space. Another direction in the investigations is to consider not only cyclic maps but also non-cyclic maps and semi-cyclic maps. In the context of best proximity points, the error estimates are found for non-cyclic contractive maps. Some results are obtained in the field of fixed points in partially ordered metric spaces. A generalization of the Ekeland variational principle related to sets generated by maps with the mixed monotone property is considered. A technique is proposed for proving results on the existence of tripled fixed points for maps with the mixed monotone property using a variational principle. Models are constructed based on actual statistical data of oligopoly markets with three participants.

INTRODUCTION

Fixed point theorems, initiated by Banach's Contraction Principle [8] has proved to be a powerful tool in nonlinear analysis.

Fixed point theory of course entails the search for a combination of conditions on a set X and a mapping $T : X \to X$ which, in turn, assures that T leaves at least one point of X fixed, i.e. $\xi = T(\xi)$ for some $\xi \in X$. Since its publication [8] there is large number of applications and generalizations.

Notations, used in the thesis

We will denote the set of natural numbers by $\mathbb{N}$ and the set of real numbers by $\mathbb{R}$. With Latin capital letters A,B,C,U,V,X,Y,Z, we will denote sets of arbitrary structure. Usually, with the letters F,G, H, and T, we will denote maps between different sets. The constants that will appear in the contractive type inequalities will be denoted by the small Greek letters α, β, γ, or the small first Latin letters a, b, and c. The Greek letter δ will be used to denote the modulus of convexity. We will denote by lowercase letters. $x, y, z, w, u, v, t, \xi, \eta, \zeta$ the elements of the considered set. We will denote by ρ the metric function defining a distance in an arbitrary set X, and by (X, ρ) we will denote a metric space, and we will denote by $(X, \|\cdot\|)$ a norm space. We prefer to use the notation ρ for the metric function instead of d, because we will use $d(U, V) = \inf\{\rho(u, v) : u \in U, v \in V\}$ for the distance between the subsets U and V of some metric space (X, ρ). We denote sometimes $\mathrm{dist(U, V)} = d(U, V)$ or even $\mathrm{dist(U, V)} = d$, just to fit some of the formulas into the text field, as far as no confusion arises.

Whenever we consider a distance ρ in a normed space, we will assume that $\rho(x, y) = \|x - y\|$.

By $X \times Y$, we will denote the Cartesian product of the sets X and Y, i.e., $u = (x, y) \in X \times X$ if $x \in X$ and $y \in Y$.

Definition 1. *Let X be a set and $T : X \to X$ be a map. A point $\xi \in X$ is said to be a fixed point for T if there holds $\xi = T\xi$.*

Fixed points in partially ordered metric spaces

We will denote by $(X, \preccurlyeq)$ a partially ordered set and by $(X, d, \preccurlyeq)$ a metric space with a partial ordering.

Definition 2. *([29, 30]) Let $(X, \preccurlyeq)$ be a partially ordered set. We say that a map $T : X \to X$ is monotone if it is either order preserving, i.e., $Tx \preccurlyeq Ty$ for all $x \preccurlyeq Y$ or order reversing, i.e., $f(x) \succcurlyeq y$ for all $x \preccurlyeq y$.*

Theorem 1. *([29]) Let $(X, d, \preccurlyeq)$ be a partially ordered complete metric spaces and $T : X \to X$ be a continuous, monotone map, such that there is $\alpha \in [0, 1)$ so that the inequality*

$$\rho(Tx, Ty) \leq \alpha\rho(x, y) \tag{1}$$

holds true for arbitrary $x, y \in X$, satisfying $x \succcurlyeq y$. A fixed point $\xi \in X$ of T exists if there is $x_0 \in X$ such that either $x_0 \preccurlyeq fx_0$ or $x_0 \succcurlyeq fx_0$.

The fixed point ξ will be unique if each pair of elements $x, y \in X$ possesses a lower bound or an upper bound.

The ideas of [29] were preceded in [23], where in partially ordered by a cone normed space $(X, \|\cdot\|)$ fixed points for monotone maps were investigated.

If inequality (1) is satisfied for all $x, y \in X$, without X being partially ordered an removing all the additional assumptions, we get the Banach's fixed points theorem, i.e., a unique fixed point for T exists.

Coupled fixed points in partially ordered metric spaces

The new notion, namely a coupled fixed point, has been introduced in [24]. Later on, results about coupled fixed points in partially ordered metric spaces have been obtained in [23].

Definition 3. *([34]) Let A be a nonempty set, and $F, G : A \times A \to A$ be two maps. If $\xi = F(\xi, \eta)$ and $\eta = G(\xi, \eta)$, then $(\xi, \eta) \in A \times A$ is said to be a coupled fixed point for the ordered pair of maps (F, G) in A.*

If $G(x, y) = F(y, x)$, we get the classical definition for coupled fixed points from [23, 24].

The mixed monotone property for a map $F : X \times X \to X$ have been generalized to a mixed monotone property for an ordered pair of maps $F, G : X \times X \to X$ in [7, 25].

Definition 4. *([7, 25]) Let $(X, \preccurlyeq)$ be a partially ordered set and let us define $F, G : X \times X \to X$. It is said (F, G) satisfies the mixed monotone property, provided that $F(x, y)$ and $G(x, y)$ are monotone nondecreasing in x and are monotone nonincreasing in y, that is, for all $x, y \in X$,*

$$F(x_1, y) \preccurlyeq F(x_2, y) \text{ and } G(x_1, y) \preccurlyeq G(x_2, y), \text{ provided that } x_1 \preccurlyeq x_2$$

and

$$F(x, y_1) \succcurlyeq F(x, y_2) \text{ and } G(x, y_1) \succcurlyeq G(x, y_2), \text{ provided that } y_1 \preccurlyeq y_2.$$

If $G(x, y) = F(y, x)$, we get the classical definition for mixed monotone property from [23, 24].

The next theorem is a generalization of the results from [24].

Theorem 2. *([7]) Let $(X, \rho, \preccurlyeq)$ be a complete metric space with a partial ordering, let $F, G : X \times X \to X$ be such that (F, G) has the mixed monotone property, and there is $\alpha \in [0, 1)$ such that*

$$\rho(F(x, y), F(u, v)) + \rho(G(x, y), G(u, v)) \leq \alpha(\rho(x, u) + \rho(y, v)) \tag{2}$$

holds for every $x \succcurlyeq u$, $y \preccurlyeq v$.

Let one of the following hold:

2.a) F and G are continuous maps

2.b) for any convergent sequence $\lim_{n\to\infty}(x_n, y_n) = (x, y)$, $(x_n, y_n) \in X \times X$

- *if $(x_n, y_n) \preccurlyeq (x_{n+1}, y_{n+1})$ then $(x_n, y_n) \preccurlyeq (x, y)$*
- *if $(x_n, y_n) \succcurlyeq (x_{n+1}, y_{n+1})$ then $(x_n, y_n) \succcurlyeq (x, y)$.*

If there are $x_0, y_0 \in X$ so that one of the following holds

- *$x_0 \preccurlyeq F(x_0, y_0)$ and $y_0 \succcurlyeq G(x_0, y_0)$,*

- $x_0 \succcurlyeq F(x_0, y_0)$ *and* $y_0 \preccurlyeq G(x_0, y_0)$,

then, a coupled fixed point $(\xi, \eta) \in X \times X$ *exists.*

If each pair of components $x, y \in X$ *also has a lower bound or an upper bound, then*

- (ξ, η) *is a unique coupled fixed point*

- *if* $G(\xi, \eta) = F(\eta, \xi)$ *then* $\xi = \eta$.

If $G(x, y) = F(y, x)$, we get the classical result from [24].

Cyclic maps

Definition 5. *([26]) Let* X *be arbitrary set and* $A, B \subset X$. *A map* $T : A \cup B \to A \cup B$ *is called a cylic map if* $T : A \to B$ *and* $T : B \to A$.

The first result about fixed points for cyclic maps is obtained in [26].

Theorem 3. *([26]) Let* A *and* B *be two nonempty closed subsets of a complete metric space, and suppose* $T : A \cup B \to A \cup B$ *be a cyclic map, such that there holds*

$$\rho(Tx, Ty) \leq \alpha \rho(x, y), \text{ for all } x \in A, y \in B, \tag{3}$$

where $\alpha \in [0, 1)$. *Then* T *has a unique fixed point in* $A \cap B$.

A direct consequence of the last theorem is that the sets A and B have a nonempty intersection. The results from [26] have been added in [28] with the error estimates and results for Kannan maps, Chaterjea maps, and Zamfirescu maps.

Unifromly Convex Banach Space

Let $(X, \|\cdot\|)$ be a normed space and denote $B_X = \{x \in X : \|x\| \leq 1\}$, $S_X = \{x \in X : \|x\| = 1\}$ the closed unit ball and the unit sphere, respectively.

Definition 6. *([12, 19]) Let* $(X, \|\cdot\|)$ *be a Banach space. For every* $\varepsilon \in (0, 2]$ *we define the modulus of convexity of* $\|\cdot\|$ *by*

$$\delta_{(X,\|\cdot\|)}(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in B_X, \|x - y\| \geq \varepsilon \right\}.$$

The norm is called uniformly convex if $\delta_{(X,\|\cdot\|)}(\varepsilon) > 0$ *for all* $\varepsilon \in (0, 2]$. *The space* $(X, \|\cdot\|)$ *is then called a uniformly convex Banach space.*

When no confusion can arise we will denote $\delta_{(X,\|\cdot\|)}$ by δ.

Definition 7. *([13], p. 42) A Banach space $(X,\|\cdot\|)$ is said to be strictly convex if $x=y$, provided $x,y\in X$ are such that $\|x\|=\|y\|=1$ and $\|x+y\|=2$.*

Best Proximity Points

If a map T is a cyclic one, then it is possible that there is no a fixed point. An idea to alter the notion of fixed point is proposed in [18], where the equality $x=Tx$ is replaced by an optimization problem $\min\{\rho(x,Tx):x\in A\cup B\}$, and T being a cyclic map.

Definition 8. *([18]) Let (X,ρ) be a metric space, A and B be subsets of X and $T:A\cup B\to A\cup B$ be a cyclic map. We say that the point $x\in A$ is a best proximity point of T in A, if $\rho(x,Tx)=\operatorname{dist}(A,B)$.*

Definition 9. *([18]) Let (X,ρ) be a metric space, A and B be subsets of X. We say that the map $T:A\cup B\to A\cup B$ is a cyclic contraction map, if it is a cyclic map and satisfies the inequality*

$$\rho(Tx,Ty)\le\alpha\rho(x,y)+(1-\alpha)\operatorname{dist}(A,B) \tag{4}$$

for some $\alpha\in(0,1)$ and every $x\in A$, $y\in B$.

Let us say that the constant $\alpha\neq 0$, because if not than (4) reduces to $\rho(Tx,Ty)\le\operatorname{dist}(A,B)$ which can hold just for some particular easy cases.

Theorem 4. *([18]) Let A and B be nonempty closed and convex subsets of a uniformly convex Banach space $(X,\|\cdot\|)$. Suppose $T:A\cup B\to A\cup B$ be a cyclic contraction map, then there exists a unique best proximity point x of T in A and Tx is a unique best proximity point of T in B.*

Error Estimate of Best proximity points in Uniformly convex Banach spaces

If $(X,\|\cdot\|)$ is a uniformly convex Banach space, then $\delta_{(X,\|\cdot\|)}(\varepsilon)$ is a strictly increasing function. Therefore if $(X,\|\cdot\|)$ is a uniformly convex Banach space then there exists the inverse function δ^{-1} of the modulus of convexity. If there exist constants $C>0$ and $q>0$, such that the inequality $\delta(\varepsilon)\ge C\varepsilon^q$ holds for every $\varepsilon\in(0,2]$ we say that the modulus of convexity is of power type q.

Theorem 5. *([33]) Let A and B be nonempty, closed and convex subsets of a uniformly convex Banach space $(X, \|\cdot\|)$, and let there exist $C > 0$ and $q \geq 2$, such that $\delta_{\|\cdot\|}(\varepsilon) \geq C\varepsilon^q$. Let $T : A \cup B \to A \cup B$ be a cyclic contraction map. Then*

5.i) there exists a unique best proximity point ξ of T in A, $T\xi$ is a unique best proximity point of T in B and $\xi = T^2\xi = T^{2n}\xi$

5.ii) for any $x_0 \in A$ the sequence $\{x_{2n}\}_{n=1}^{\infty}$ converges to ξ and $\{x_{2n+1}\}_{n=1}^{\infty}$ converges to $T\xi$, where $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$

5.iii) a priori error estimate holds

$$\left\|\xi - T^{2n}x_0\right\| \leq \frac{\|x_0 - Tx_0\|}{1 - \sqrt[q]{\alpha^2}} \sqrt[q]{\frac{\|x_0 - Tx_0\| - d}{Cd}} \left(\sqrt[q]{\alpha}\right)^{2n} \tag{5}$$

5.iv) a posteriori error estimate holds

$$\left\|T^{2n}x_0 - \xi\right\| \leq \frac{\|T^{2n-1}x_0 - T^{2n}x_0\|}{1 - \sqrt[q]{\alpha^2}} \sqrt[q]{\frac{\|T^{2n-1}x_0 - T^{2n}x_0\| - d}{Cd}} \sqrt[q]{\alpha}, \tag{6}$$

where $d = \operatorname{dist}(A, B)$.

Reflexive Banach Spaces

Definition 10. *([19]) Let $(X, \|\cdot\|)$ be a normed space. By X^* we denote the vector space of all continuous linear functionals on X, endowed with the norm $\|f\|^* = \sup\{|f(x)| : x \in B_X\}$, called the canonical dual norm. The space $(X^*, \|\cdot\|^*)$ is called the dual space of X.*

The space X^{**} is the dual space of X^*. The mapping $\pi : X \to X^{**}$ defined by $\pi(x) = \psi_{x_0}(f) = f(x_0)$ maps the set X onto some subset $\pi(X) \subseteq X^{**}$. The map π is called the natural mapping of X onto X^{**}.

Definition 11. *([19]) Let $(X, \|\cdot\|)$ be a normed space. If $\pi(X) \equiv X^{**}$, then X is called reflexive.*

Best Proximity Points in Reflexive Banach Spaces

The first result about existence and uniqueness of best proximity points for cyclic maps $T : A \cup B \to A \cup B$ in reflexive Banach space is in [6].

Theorem 6. *([19]) A sequence $\{x_n\}_{n=1}^{\infty}$ of elements in a normed space $(X, \|\cdot\|)$ is weakly convergent to an element $x \in X$ if and only if the numerical sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x)$ for every $f \in X^*$ and is denoted by $w \lim\limits_{n\to\infty} x_n = x$.*

Definition 12. *([19]) Let $(X, \|\cdot\|)$ be a normed space. A set $A \subset X^*$ is called a weakly closed one if for whenever thereholds $w \lim\limits_{n\to\infty} x_n = x$, with $x_n \in A$, then there holds $x \in A$.*

Theorem 7. *Let A and B be nonempty weakly closed subsets of a reflexive Banach space X and let $T : A \cup B \to A \cup B$ be a cyclic contraction map. Then there exists $(x, y) \in A \times B$ such that $\|x - y\| = \operatorname{dist}(A, B)$.*

Let us say that although we assume the sets to be weakly closed, we can not verify that $y = Tx$.

Definition 13. *Let A and B be nonempty subsets of a normed space $(X, \|\cdot\|)$, $T : A \cup B \to A \cup B$ be a cyclic map. We say that T satisfies the proximal property if $\|x - Tx\| = \operatorname{dist}(A, B)$ holds whenever there hold $w \lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = \operatorname{dist}(A, B)$.*

Theorem 8. *Let A and B be nonempty subsets of a reflexive Banach space $(X, \|\cdot\|)$ such that A is weakly closed and let $T : A \cup B \to A \cup B$ be a cyclic contraction map. Then there exists $x \in A$ such that $\|x - Tx\| = \operatorname{dist}(A, B)$, provided one of the following conditions is satisfied:*

8.a) T is weakly continuous on A

8.b) T satisfies the proximal property.

If in addition $(X, \|\cdot\|)$ be a strictly convex, then the best proximity point x is unique and there holds $T^2x = x$.

Noncyclic Maps

Definition 14. *([1]) Let X be arbitrary set and $A, B \subset X$. A map $T : A \cup B \to A \cup B$ is called a noncylic map if $T : A \to A$ and $T : B \to B$.*

Now let $T : A \cup B \to A \cup B$ be a noncyclic mapping. We can consider the following minimization problem: $\min\{\rho(x, Tx) : x \in A\}$, $\min\{\rho(y, Ty) : y \in A\}$, and $\min\{\rho(x, y) : (x, y) \in A \times B\}$.

Best Proximity Pair for a Noncyclic Map

Definition 15. *([1]) Let (X, ρ) be a metric space and $A, B \subset X$ and T be a noncyclic map. An ordered pair $(\xi, \eta) \in A \times B$ is called be a best proximity pair of fixed points for the noncyclic mapping T provided that $\xi = T\xi$, $\eta = T\eta$, and $d(\xi, \eta) = \operatorname{dist}(A, B)$.*

Some times a best proximity pair for the noncyclic mapping T is called an optimal pair of fixed points of the noncyclic map T.

A natural generalization seem to be the search for conditions that guarantee existence and uniqueness of best proximity pairs for noncyclic maps.

Definition 16. *([22]) Let (X, ρ) be a metric space and $A, B \subset X$. A noncyclic map $T : A \cup B \to A \cup B$ is called a noncyclic contraction map if there exists $\alpha \in [0, 1)$ so that the inequality $\rho(Tx, Ty) \le \alpha\rho(x, y) + (1 - \alpha)\operatorname{dist}(A, B)$ holds for every $x \in A$ and $y \in B$.*

Theorem 9. *([20]) Let A and B be two closed convex subsets of a strictly convex and reflexive Banach space $(X, \|\cdot\|)$. Suppose that $T : A \cup B \to A \cup B$ is a noncyclic contraction map. Then T has a best proximity pair.*

Ordered Pairs of Cyclic Maps

Definition 17. *([34]) Let A_x, A_y, B_x and B_y be nonempty subsets of X. Let $F : A_x \times A_y \to B_x$, $f : A_x \times A_y \to B_y$, $G : B_x \times B_y \to A_x$ and $g : B_x \times B_y \to A_y$. For any pair $(x, y) \in A_x \times A_y$ we define the sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ by $x_0 = x$, $y_0 = y$ and*

$$\begin{aligned} x_{2n+1} &= F(x_{2n}, y_{2n}), & y_{2n+1} &= f(x_{2n}, y_{2n}) \\ x_{2n+2} &= G(x_{2n+1}, y_{2n+1}), & y_{2n+2} &= g(x_{2n+1}, y_{2n+1}) \end{aligned}$$

for all $n \ge 0$.

Lower semi-continuous maps

Let (X, ρ) be a metric space. Following [10] an extended real valued function $T : X \to (-\infty, +\infty]$ on X is called lower semicontinuous (for short l.s.c) if $\{x \in X : f(x) > a\}$ is an open set for each $a \in (-\infty, +\infty]$. Equivalently T is l.s.c if and only if at any point $x_0 \in X$ there holds $\liminf\limits_{x \to x_0} f(x) \ge f(x_0)$. A function T is called to be proper function, provided that $T \not\equiv +\infty$.

Ekeland's Variational Principle and Fixed Point Results

Ekeland proved a variational principle in [16]. In a series of articles he enriches the results. Later he presented a more concise proof [17], which technique we will use. In the same article [17], the Banach's fixed point theorem is proven with the help of the variational principle.

Theorem 10. *([17]) Let (X, ρ) be a complete metric space, and $T : X \to \mathbb{R} \cup \{+\infty\}$ be a proper l.s.c. function, bounded from below. Let $\varepsilon > 0$ be given, and a point $u_0 \in X$ be such that $T(u_0) \leq \inf_{v \in X} T(v) + \varepsilon$.*

Then there exists some point $v_0 \in X$ such that $T(v_0) \leq T(u_0)$, $\rho(u_0, v_0) \leq 1$, and for every $w \in X$ different from v_0 holds the inequality $T(w) > T(v_0) - \varepsilon\rho(w, v_0)$.

Chapter I
Coupled Best Proximity Points for Cyclic, Semi-Cyclic, and Non-Cyclic Maps in Banach Spaces

The results in this chapter are based on the publications of the author [2, 4].

Coupled Best Proximity Points for Cyclic Maps in Reflexive Banach Spaces

We will start first with the results from [4].

Definition 18. *Let $(X, \|\cdot\|)$ be a Banach space and $U, V \subset X$ be subsets. We say that a map $F : U \times V \to X$ is weakly continuous if for any weakly convergent sequences $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$, $u_n \in U$ and $v_n \in V$ there holds $w\lim_{n\to\infty} F(u_n, v_n) = F(u, v)$, where $u = w\lim_{n\to+\infty} u_n$ and $v = w\lim_{n\to+\infty} v_n$.*

.

Definition 19. *Let $(X, \|\cdot\|)$ be a Banach space and $A_x, A_y, B_x, B_y \subset X$ be subsets and $F : A_x \times A_y \to B_x$, $f : A_x \times A_y \to B_y$. We say that the pair of maps (F, f) satisfies the proximal property if for any weakly convergent sequences $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$, $u_n \in A_x$ and $v_n \in A_y$ such that $u = w\lim_{n\to+\infty} u_n$, $v = w\lim_{n\to+\infty} v_n$ whenever there hold*

$$\lim_{n\to+\infty} \|u_n - F(u_n, v_n)\| = \operatorname{dist}(A_x, B_x) \text{ and } \lim_{n\to+\infty} \|v_n - f(u_n, v_n)\| = \operatorname{dist}(A_y, B_y),$$

then hold $\|u - F(u, v)\| = \operatorname{dist}(A_x, B_x)$ and $\|v - f(u, v)\| = \operatorname{dist}(A_y, B_y)$.

Definition 20. *Let (X,ρ) be a metric space, $A_x, A_y, B_x, B_y \subset X$ be non-empty, proper subsets, $F : A_x \times A_y \to B_x$, $f : A_x \times A_y \to B_y$. An ordered pair $(x,y) \in A_x \times A_y$ is called a coupled best proximity point of (F,f) in $A_x \times A_y$ if $\rho(x, F(x,y)) = \operatorname{dist}(A_x, B_x)$ and $\rho(y, f(x,y)) = \operatorname{dist}(A_y, B_y)$.*

Definition 21. *Let A_x, A_y, B_x, and B_y be non-empty subsets of a metric space (X,ρ), $F : A_x \times A_y \to B_x$, $f : A_x \times A_y \to B_y$, $G : B_x \times B_y \to A_x$ and $g : B_x \times B_y \to A_y$. The ordered pair of ordered pairs $((F,f),(G,g))$ is said to be a cyclic contraction ordered pair if there are reals $\alpha, \beta, \gamma, \delta \geq 0$, so that $\max\{\alpha+\gamma, \beta+\delta\} < 1$ and there holds the inequality*

$$
\begin{aligned}
S &= \rho(F(x,y), G(u,v)) + \rho(f(z,w), g(t,s)) \\
&\leq \alpha\rho(x,u) + \beta\rho(y,v) + \gamma\rho(z,t) + \theta\rho(w,s) \\
&\quad +(1-(\alpha+\beta))\operatorname{dist}(A_x,B_x) + (1-(\beta+\theta))\operatorname{dist}(A_y,B_y)
\end{aligned}
\tag{7}
$$

for any $(x,y),(z,w) \in A_x \times A_y$ and $(u,v),(t,s) \in B_x \times B_y$.

Definition 22. *Let A_x, A_y, B_x and B_y be nonempty subsets of X. Let $F : A_x \times A_y \to B_x$, $f : A_x \times A_y \to B_y$, $G : B_x \times B_y \to A_x$ and $g : B_x \times B_y \to A_y$. For any initial, arbitrary chosen $(\xi,\eta) \in A_x \times A_y$, we define the sequences $\{\xi_n\}_{n=0}^{\infty}$ and $\{\eta_n\}_{n=0}^{\infty}$ by $\xi_0 = \xi$, $\eta_0 = \eta$ and*

$$
\begin{array}{ll}
\xi_{2n+1} = F(\xi_{2n}, \eta_{2n}), & \eta_{2n+1} = f(\xi_{2n}, \eta_{2n}), \\
\xi_{2n+2} = G(\xi_{2n+1}, \eta_{2n+1}), & \eta_{2n+2} = g(\xi_{2n+1}, \eta_{2n+1})
\end{array}
$$

for all $n \geq 0$.

Theorem 11. *Let $(X, \|\cdot\|)$ be a reflexive Banach space, $A_x, A_y, B_x, B_y \subset X$ be nonempty, proper, weakly closed sets of X, $F : A_x \times A_y \to B_x$, $f : A_x \times A_y \to B_y$, $G : B_x \times B_y \to A_x$ and $g : B_x \times B_y \to A_y$. Let $((F,f),(G,g))$ be a cyclic contraction of ordered pairs. Let there hold one of the following*

11.a) F and f be weakly continuous on $A_x \times A_y$ and G and g be weakly continuous on $B_x \times B_y$

11.b) (F,f) satisfies the proximal property.

Then (F,f) has a coupled best proximity point $(\xi_x, \xi_y) \in A_x \times A_y$ and (G,g) has a coupled best proximity point $(\eta_x, \eta_y) \in B_x \times B_y$, i.e.,

$$
\|\xi_x - F(\xi_x, \xi_y)\| = \operatorname{dist}(A_x, B_x), \quad \|\xi_y - f(\xi_x, \xi_y)\| = \operatorname{dist}(A_y, B_y)
$$

and

$$
\|\eta_x - G(\eta_x, \eta_y)\| = \operatorname{dist}(A_x, B_x), \quad \|\eta_y - g(\eta_x, \eta_y)\| = \operatorname{dist}(A_y, B_y).
$$

If, in addition $(X, \|\cdot\|)$ *is a strictly convex Banach space and* $A_x, A_y, B_x, B_y \subset X$ *are convex subsets, then*

$$\xi_x = G(F(\xi_x, \xi_y), f(\xi_x, \xi_y)), \quad \xi_y = g(F(\xi_x, \xi_y), f(\xi_x, \xi_y))$$

and

$$\eta_x = F(G(\eta_x, \eta_y), g(\eta_x, \eta_y)), \quad \eta_y = f(F(\eta_x, \eta_y), f(\eta_x, \eta_y)).$$

If $A_x = A_y$*,* $B_x = B_y$*,* $f(x, y) = F(y, x)$ *and* $g(x, y) = G(y, x)$*,* $(X, \|\cdot\|)$ *is a strictly convex Banach space then the coupled fixed point* $(\xi_x, \xi_y) \in A_x \times A_x$ *satisfies* $\xi_x = \xi_y$*.*

Coupled Best Proximity Points for Semi-Cyclic Maps in Reflexive Banach Spaces

Definition 23. *Let* $(X, \|\cdot\|)$ *be a Banach space and* $U, V \subset X$ *be subsets and* $F : U \times V \to U$*,* $G : U \times V \to V$*. We say that the pair of maps* (F, G) *satisfies the proximal property if for any weakly convergent sequences* $\{u_n\}_{n=0}^{\infty}$ *and* $\{v_n\}_{n=0}^{\infty}$*,* $u_n \in U$ *and* $v_n \in V$ *such that* $u = w\lim_{n\to+\infty} u_n$*,* $v = w\lim_{n\to+\infty} v_n$ *whenever it holds*

$$\lim_{n\to+\infty} \|u_n - G(u_n, v_n)\| = \operatorname{dist}(U, V) \text{ and } \lim_{n\to+\infty} \|v_n - F(u_n, v_n)\| = \operatorname{dist}(U, V)$$

there hold $\|u - G(u, v)\| = \|v - F(u, v)\| = \operatorname{dist}(U, V)$*.*

Definition 24. *Let* A, B *be subsets of a metric space* (X, ρ)*. An ordered pair of maps* (F, G) $F : A \times B \to A$*,* $G : A \times B \to B$ *will be called a semi-cyclic contraction if there exists* $\alpha \in [0, 1/2)$*, such that for any* $(x, y), (u, v) \in A \times B$ *there holds the inequality*

$$\rho(F(x, y), G(u, v)) - \operatorname{dist}(\mathrm{A}, \mathrm{B}) \le \alpha(\rho(x, v) + \rho(y, u) - 2\operatorname{dist}(\mathrm{A}, \mathrm{B})). \tag{8}$$

Theorem 12. *Let* $(X, \|\cdot\|)$ *be a reflexive Banach space and* $A, B \subset X$*, be nonempty, proper, weakly closed sets, and* $F : A \times B \to A$*,* $G : A \times B \to B$ *be a semi-cyclic contraction. Let there hold one of the following*

12.a) F *and* G *are weakly continuous on* $A \times B$

12.b) (F, G) *satisfy the proximal property.*

Then there exists $(\xi, \eta) \in A \times B$*, which is a best proximity point of* (F, G)*.*

If in addition $(X, \|\cdot\|)$ *is a strictly convex Banach space and* $A, B \subset X$ *are convex subsets, then* $\xi = F(F(\xi, \eta), G(\xi, \eta))$*,* $\eta = G(F(\xi, \eta), G(\xi, \eta))$*, and* $\xi = F(\xi, \eta)$*,* $\eta = G(\xi, \eta)$*.*

Coupled Best Proximity Points for Noncyclic Maps in Uniformly Convex Banach Space

This section is based on [2].

We will generalize the ideas of noncyclic maps and optimal points [1], best proximity points of maps of two variables [32] to introduce the notion of optimal points for noncyclic maps of two variables.

Definition 25. *Let (Z,ρ) be a metric space. Let $X, Y \subset Z$ be sets, such that $\mathrm{dist}(X,Y) > 0$. Let $F : X \times X \to X$ and $G : Y \times Y \to Y$. We will call the ordered pair of maps (F,G) a noncyclic maps of two variables. An ordered pair of ordered pairs $((x,y),(u,v) \in (X \times X) \times (Y \times Y)$ is called an optimal pair of coupled fixed points of the ordered pair of noncylic maps (F,G), provided that (x,y) is a coupled fixed point of F, (u,v) is a coupled fixed point of G and $\rho(x,u) = \rho(y,v) = \mathrm{dist}(A,B)$.*

Definition 26. *Let (Z,d) be a metric space. Let $X, Y \subset Z$ be sets, such that $\mathrm{dist}(X,Y) > 0$. Let $F : X \times X \to X$ and $G : Y \times Y \to Y$. We say that the ordered pair of noncylic maps (F,G) is a noncyclic contraction if there is $\alpha \in [0,1)$ so that the inequality $\|F(x,y) - G(u,v)\| \le \alpha\|x-u\| + (1-\alpha)\mathrm{dist}(X,Y)$ holds for any $x, y \in X$ and $u, v \in Y$.*

Theorem 13. *Let $(Z, \|\cdot\|)$ be a uniformly convex Banach space. Let $X, Y \subset Z$ be two convex sets, such that $\mathrm{dist}(X,Y) > 0$. Let $F : X \times X \to X$ and $G : Y \times Y \to Y$ be such that the ordered pair of maps (F,G) be a noncyclic contraction map. Then*

13.i) the ordered pair of maps (F,G) has a unique optimal pair $((x,y),(u,v))$ of coupled fixed points $(x,y) \in X \times X$ and $(u,v) \in Y \times Y$

13.ii) for every $(x_0,y_0) \in X \times X$, $(u_0,v_0) \in Y \times Y$ the sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ converge to x, y, u and v, respectively

13.iii) a priori error estimate holds

$$\max\{\|x-x_m\|, \|y-y_m\|\} \le \max\{M_0,N_0\}\sqrt[q]{\frac{\max\{M_0,N_0\}-d}{Cd}}\frac{\sqrt[q]{\alpha^m}}{1-\sqrt[q]{\alpha}}$$

13.iv) a posteriori error estimate holds

$$\max\{\|x_n-x\|, \|y_n-y\|\} \le \frac{\max\{M_n,N_n\}}{1-\sqrt[q]{\alpha}}\sqrt[q]{\frac{\max\{M_n,N_n\}-d}{Cd}},$$

where for every $x \in A$ *and* $y \in B$ *we use the notation*

$$M_n = \max\{\|x_n - u_n\|, \|F(x_n, y_n) - u_n\|\},$$

$$N_n = \max\{\|y_n - v_n\|, \|F(y_n, x_n) - v_n\|\},$$

and $d = \operatorname{dist}(X, Y)$.

If in addition on of the sets X *or* Y *is strictly convex then* $x = y$ *and* $u = v$.

Chapter II
Variational Principles in Partially Ordered metric Spaces

The results in this chapter are from [3, 5].

Hardy-Roger's Maps with the Mixed Monotone Property in Partially Ordered Metric Spaces

The results in this section are from [5].

Theorem 14. *Let* $(X, \rho, \preceq)$ *be a partially ordered complete metric space and* $F : X \times X \to X$ *be a continuous map with the mixed monotone property. Let there exist* $\alpha + \beta + \gamma \in [0, 1/2)$, *so that the inequality*

$$\begin{array}{rcl} \rho(F(x,y), F(u,v)) & \leq & \alpha(\rho(x,u) + \rho(y,v)) \\ & & +\beta(\rho(x, F(x,y) + \rho(u, F(u,v)) \\ & & +\gamma(\rho(x, F(u,v)) + \rho(u, F(x,y)). \end{array} \tag{9}$$

holds for all $x \succeq u$ *and* $y \preceq v$. *If there exists at least one ordered pair* (x, y), *such that* $x \preceq F(x, y)$ *and* $y \succeq F(y, x)$, *then there exists a coupled fixed point* (x, y) *of* F.

If in addition every pair of elements in $X \times X$ *has an lower or an upper bound, then the coupled fixed point is unique.*

Tripled fixed points for map with the mixed monotone property

This section is based on the results from [3].

Let $(X, \preccurlyeq)$ be a partially ordered set. Following [9], whenever we consider the Cartesian product space $X \times X \times X = X^3$ we will endow it with the following partial order $(u, v, w) \preceq (x, y, z)$ if $x \preccurlyeq u$, $y \preccurlyeq v$ and $z \preccurlyeq w$, provided that $(x, y, z), (u, v, w) \in X \times X \times X$.

Let (X, ρ) be a metric space. Following [9], whenever we consider the Cartesian product space $X \times X \times X$ we will endow it with the metric

$$\rho_1((x, y, z), (u, v, w)) = \rho(x, u) + \rho(y, v) + \rho(z, w),$$

provided that $(x, y, z), (u, v, w) \in X \times X \times X$.

Definition 27. *Let X be a set and $F : X \times X \times X \to X$, $G : X \times X \times X \to X$, and $H : X \times X \times X \to X$. The ordered triple (F, G, H) is called and ordered triple of maps.*

Definition 28. *Let X be a set and (F, G, H) be an ordered triple of maps. An element $(x, y, z) \in X \times X \times X$ is called triple fixed point of if*

$$\left|\begin{array}{l} \xi = F(\xi, \eta, \zeta) \\ \eta = G(\xi, \eta, \zeta) \\ \zeta = H(\xi, \eta, \zeta). \end{array}\right.$$

If $G(x, y, z) = F(y, x, y)$ and $H(x, y, z) = F(z, y, x)$ we get the tripled fixed points introduced and investigated in [9].

Definition 29. *Let $(X, \preccurlyeq)$ be a partially ordered set and (F_1, F_2, F_3) be an ordered triple of maps. We say that (F_1, F_2, F_3) has the mixed monotone property if for any $x, x_1, x_2, y, y_1, y_2, z, z_1, z_2 \in X$ there hold*

$$F_i(x_1, y, z) \preccurlyeq F_i(x_2, y, z), \quad \textit{holds for every } \ i = 1, 2, 3, \ \textit{provided that } x_1 \preccurlyeq x_2$$

$$F_i(x, y_1, z) \succcurlyeq F_i(x, y_2, z), \quad \textit{holds for every } \ i = 1, 2, 3, \ \textit{provided that } y_1 \preccurlyeq y_2$$

and

$$F_i(x, y, z_1) \preccurlyeq F_i(x, y, z_2), \quad \textit{holds for every } \ i = 1, 2, 3, \ \textit{provided that } z_1 \preccurlyeq z_2.$$

If $G(x, y, z) = F(y, x, y)$ and $H(x, y, z) = F(z, y, x)$ we get notion of for a map F, introduced in [9].

Theorem 15. *Let $(X, \rho, \preccurlyeq)$ be a partially ordered complete metric space and (F, G, H) be an ordered triple of continuous maps.*

Let us put $\overline{x} = (x^{(1)}, x^{(2)}, x^{(3)}) \in X \times X \times X$ and consider the subset $Y^3 \subset X^3$ defined by

$$Y^3 = \{(x^{(1)}, x^{(2)}, x^{(3)}) \in X^3 : x^{(1)} \preccurlyeq F(\overline{x}), \ x^{(2)} \succcurlyeq G(\overline{x}) \textit{ and } x^{(3)} \preccurlyeq H(\overline{x})\} \neq \emptyset.$$

Let $T : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, l.s.c., bounded from below function. Let for any $\varepsilon > 0$ be arbitrary chosen and fixed there is a $\overline{u}_0 \in Y^3$, such that the inequality $T(\overline{u}_0) \leq \inf\limits_{\overline{v} \in Y^3} T(\overline{v}) + \epsilon$ holds.

Then therefore exists an ordered triple $\overline{x} \in Y^3$, such that $T(\overline{x}) \leq T(\overline{u}_0)$, $\rho_1(\overline{x}, \overline{u}_0) \leq 1$, and for every $\overline{w} \in Y^3$ different from $\overline{x} \in Y^3$ holds the inequality

$$T(\overline{w}) > T(\overline{x}) - \epsilon d(\overline{w}, \overline{x}).$$

Applications of Theorem 15

Theorem 16. *Let $(X, \rho, \preccurlyeq)$ be a partially ordered complete metric space, and (F_1, F_2, F_3) be an ordered triple of continuous maps with the mixed monotone property. Let there exists $\alpha \in [0, 1)$, so that the inequality*

$$\sum_{k=1}^{3} \rho(F_k(x, y, z), F_k(u, v, w)) \leq \alpha(\rho(x, u) + \rho(y, v) + \rho(z, w))$$

holds for all $x \succcurlyeq u$, $y \preccurlyeq v$ and $z \succcurlyeq w$. If there exists at least one ordered pair (x, y, z), such that $x \preccurlyeq F_i(x, y, z)$, $y \succcurlyeq F_2(x, y, z)$ and $z \preccurlyeq F_3(x, y, z)$, then there exists a tripled fixed points (x, y, z) of (F_1, F_2, F_3).

If in addition every pair of elements in X^3 has an lower or an upper bound, then the tripled fixed point is unique.

The next theorem, which is a corollary of Theorem 16 covers a wide class of the investigated maps from [9].

Theorem 17. *([9]) Let $(X, \rho, \preccurlyeq)$ be a complete partially ordered metric space and F be a continuous maps with the mixed monotone property. Let there exist $\alpha, \beta, \gamma \in [0, 1)$, satisfying $s = \alpha + 2\beta + \gamma < 1$, so that the inequality*

$$\rho(F(x, y, z), F(u, v, w)) \leq \alpha\rho(x, u) + \beta\rho(y, v) + \gamma\rho(z, w) \tag{10}$$

holds for all $x \succcurlyeq u$, $y \preccurlyeq v$, and $z \succcurlyeq w$. If there exists at least one ordered tripled (x, y, z), such that $x \preccurlyeq F(x, y, z)$, $y \succcurlyeq F(y, x, y)$, and $z \preccurlyeq F(z, y, x)$, then there exists a tripled fixed points (x, y, z) of F, i.e., $x = F(x, y, z)$, $y = F(y, x, y)$ and $z = F(z, y, x)$.

If in addition each pair of elements in $X \times X \times X$ has an lower or an upper bound, then the tripled fixed point is unique.

Chapter III
Applications in Market Equilibrium of Oligopoly Markets

We would like to introduce the main concepts and results in the field of equilibrium theory in oligopoly markets.

Let us first start with an oligopoly model where n companies are competing for the same consumers for just one good and striving to meet the demand with an overall production of $Z = \sum_{k=1}^{n} x_k$, where x_k is the production of the k-th player [21, 31]. The market price is defined as $P(Z) = P\left(\sum_{k=1}^{n} x_k\right)$, which is the inverse of the demand function. Market players have cost functions $c_k(x)$, $k = 1, 2, \ldots, n$, respectively. $\Pi_k(x_1, x_2, \ldots, x_n) = x_k P\left(\sum_{k=1}^{n} x_k\right) - c_k(x_k)$ for $k = 1, 2, \ldots, n$ be the payoff function of the k-th player.

Assuming that firms are acting rationally, i.e., the goal of each company is to maximize its profit, assuming the the other players do not change their levels of production, we have

$$(11) \qquad \left| \begin{array}{l} \max\left\{\Pi_k(x_1, x_2, \ldots, x_n) : x_k, \text{ assuming that } x_j, j \neq k \text{ is fixed}\right\} \\ k = 1, 2, \ldots, n. \end{array} \right.$$

Provided that functions P and c_k, $k = 1, 2, \ldots n$ are differentiable, we get the equations

$$(12) \qquad \left| \begin{array}{l} \frac{\partial \Pi_i(x_1, x_2, \ldots, x_n}{\partial x_i} = P(\sum_{j=1}^{n} x_j) + x_i P'(\sum_{j=1}^{n} x_j) - c_i'(x_i) = 0 \\ k = 1, 2, \ldots, n. \end{array} \right.$$

The solution to the maximization of the payoff functions Π_k for $k = 1, 2, \ldots n$ are the solutions of the first-order system of equations (12). To be sure that a solution of (12) presents the equilibrium of production in the considered oligopoly market [21, 31], i.e., the solution of the maximization of the payoff functions Π_k for $k = 1, 2, \ldots n$, some additional assumptions should be checked. It is either the payoff functions Π_k for $k = 1, 2, \ldots n$ being concave or the solution $(\xi_1, \xi_2, \ldots, \xi_n)$ should satisfy the second order conditions $\frac{\partial^2 \Pi_k(x_1, x_2, \ldots, x_n)}{\partial x_k^2}(\xi_1, \xi_2, \ldots, \xi_n) < 0$ for $k = 1, 2, \ldots n$.

If we stick to a duopoly market, i.e., a market with two producers, often equations (12) have solutions in the form of $x_1 = b_1(x_2)$ and $x_2 = b_2(x_1)$, which are called response functions ([21]). The same response functions can arise, and in the case of three players on the market, i.e., $x_1 = b_1(x_2, x_3)$, $x_2 = b_2(x_1, x_3)$, and $x_3 = b_3(x_1, x_2)$ [21, 31].

It may turn out difficult or impossible to solve (12); thus, it is often advised to search for an approximate solution. Another drawback when searching for an approximate solution is that it may not be stable. Fortunately, we can find an implicit formula for the response function in (12), i.e.,

$$x_k = \frac{\partial \Pi_k(x_1, x_2, \ldots, x_n)}{\partial x_k} + x_k = G_k(x_1, x_2, \ldots, x_n)$$

for $k = 1, 2, \ldots n$. Thus we get the notions of generalized coupled fixed points $x_1 = G_1(x_1, x_2)$ and $x_2 = G_2(x_1, x_2)$ in the case of a duopoly market and generalized tripled fixed points $x_1 = G_1(x_1, x_2, x_3)$, $x_2 = G_2(x_1, x_2, x_3)$ and $x_3 = G_3(x_1, x_2, x_3)$ if three players are involved in the market.

Application of Theorem 12 in Market Equilibrium in Duopoly Markets

The next example is similar to that in [15], except that the norm, instead of the Euclidean one $\|\cdot\|$, is the summing one $\|\cdot\|_1$.

Example 1. *Let us consider a market with two competing firms; each firm produces two products, and any one of the items is completely replaceable with a similar product of the other firm. Let us assume that the first firm can produce much fewer quantities than the second one, i.e., if* x_1, x_2 *are the quantities produced by the first firm and* y_1, y_2 *are the quantities produced by the second one, then* $x_1, x_2 \in [0, 1]$ *and* $y_1, y_2 \in [2, 3]$. *Let* $A = [0, 1] \times [0, 1]$ *and* $B = [2, 3] \times [2, 3]$ *be considered as subsets of* $(\mathbb{R}^2, \|\cdot\|_1)$, *which is a Banach space. Let us consider the response functions* $F(x_1, x_2, y_1, y_2)$ *and* $f(x_1, x_2, y_1, y_2)$ *defined by*

$$F(x,y) = \begin{cases} \frac{3x_1}{8} + \frac{x_2}{8} - \frac{3y_1}{16} - \frac{y_2}{16} + 1 \\ \frac{x_1}{8} + \frac{3x_2}{8} - \frac{y_1}{16} - \frac{3y_2}{16} + 1 \end{cases}, \quad f(x,y) = \begin{cases} -\frac{3x_1}{16} - \frac{x_2}{16} + \frac{3y_1}{4} + \frac{y_2}{4} + \frac{5}{4} \\ -\frac{x_1}{16} - \frac{3x_2}{16} + \frac{y_1}{4} + \frac{3y_2}{4} + \frac{5}{4} \end{cases}.$$

The ordered pair (F, f) satisfies Theorem 12. Thus there exists an equilibrium pair $(x, y) = ((x_1, x_2), (y_1, y_2))$ and for any initial start in the economy, the iterated sequence $(x^n, y^n) = ((x_1^n, x_2^n), (y_1^n, y_2^n))$ converges to the market equilibrium (x, y). We get in this case that the equilibrium pair of the production of the two firms is $x = (1, 1)$, $y = (2, 2)$, and the total production will be $a = (3, 3)$.

This example shows that the geometry of the underlying space, the geometry of the domain $A \times B$, and the positioning of the subsets A and B are crucial for the uniqueness of the existing coupled best proximity points.

Some statistical measurments in modeling of data

Investigating the stationarity of time series is an important factor for our study. A stationary time series is a series whose statistical measures, such as mean, variance, covariance, and standard deviation, are not a function of time. In other words, stationarity in a time series means a time series without trends or seasonal components. The two most popular statistical tests to ascertain whether or not a time series is stationary are the augmented Phillips-Perron (PP) and Dickey-Fuller (ADF) tests [14]. The Dickey-Fuller (ADF) test evaluates the hypothesis of a unit root $\alpha_1 = 1$ in the model $Y_t = \alpha_1 Y_{t-1} + \epsilon_t$, or, equivalently, the hypothesis of $\gamma = 0$ in $\Delta Y_t = \gamma Y_{t-1} + \epsilon_t$, where Y_t denotes the value of a data point at period t, $\Delta Y_t = Y_t - Y_{t-1}$, ϵ_t – a pure random disturbance in t. Tests involving lagged changes are called augmented Dickey-Fuller tests and are also used for hypothesis testing $\gamma = 0$. The null hypothesis H_0 means that the time series is non-stationarity and indicates the presence of a trend; the alternative hypothesis H_1 means stationarity. Put otherwise, the small p-value indicates that it is improbable that a unit root exists.

The least squares method

The Least Squares Method (LSM) is one of the most widely used methods in statistics and holds a special place in the development of science. Adrien-Marie Legendre [27] initially published the LSM in 1805, which was established by Carl Friedrich Gauss in 1795. The method is usually associated with linear statistical models, where its efficacy is most evident.

When assuming a linear form of dependence, we can written as: $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_k x_k + \epsilon$, where the coefficients $\beta_0, \beta_1, \beta_2, \ldots, \beta_k$ are the ones we are searching for.

For n joint observations of the dependent y and independent variable X is expressed in the following matrix form: $Y = X\beta + \epsilon$, where $Y = (y_1, y_2, \ldots, y_n)^T$ is the variable observation vector of $n \times 1$, X is the known matrix of $n \times (k+1)$ with the first column consisting of ones, $\beta = (\beta_0, \beta_1, \ldots, \beta_k)^T$ is the unknown parameter vector of $(k+1) \times 1$, and $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)^T$ of $n \times 1$ is the error vector. Finding the parameters β so that the sum of the squares of the errors is as little as possible is the basic goal of the least squares approach [11], i.e., $(Y - X\beta)^T (Y - X\beta) \to \min$.

Statistical measures for model estimation

We employ two basic statistical metrics to assess the effectiveness of the built models: the mean absolute percentage error $MAPE = \frac{1}{N} \sum_{t=1}^{n} \left| \frac{P_t - Y_t}{Y_t} \right|$ and the coefficient of determination $R^2 = 1 - \frac{\sum_{t=1}^{n} (P_t - Y_t)^2}{\sum_{t=1}^{n} (Y_t - \overline{Y})^2}$, where the sample size is n, the predicted values are P_t, the mean is $\overline{Y}$, and the values of the dependent variable are Y_t. Our objective is to develop a model with the lowest MAPE and the maximum potential value of R^2.

Application of Theorem 16 in Investigation of Market Equilibrium of Tripodal markets

We will use an easy-to-implement version of Theorem 16.

Corollary 1. *([3]) Let us consider an oligopoly market with three players, satisfying:*

1. *the three players are producing perfect substitutes of a homogeneous good*
2. *the i-th player can produce quantities from the set X_i, $i = 1, 2, 3$, where X_i are closed nonempty subsets of a complete metric space $(\mathbb{R}, |\cdot|)$*
3. *let the ordered tripled $(F_1(x, y, z), F_2(x, y, z), F_3(x, y, z))$, $F_i : X_1 \times X_2 \times X_3 \to X_i$, $i = 1, 2, 3$ be a semi-cyclic map that presents the response functions for players one, two, and three, respectively*
4. *(F_1, F_2, F_3) be an ordered triple of continuous maps with the mixed monotone property*
5. *let there be $0 < \alpha < 1$, so that the inequality:*

$$\sum_{i=1}^{3} |F_i(x, y, z) - F_i(u, v, w)| \leq \alpha(|x - u| + |y - v| + |z - w|) \tag{13}$$

holds for all $x \succcurlyeq u$, $y \preccurlyeq v$, and $z \succcurlyeq w$

6. *there exists at least one ordered tripled (x, y, z), such that $x \preccurlyeq F_i(x, y, z)$, $y \succcurlyeq F_2(x, y, z)$ and $z \preccurlyeq F_3(x, y, z)$.*

Then there exists an ordered tripled (x, y, z) which is a market equilibrium, i.e., a tripled fixed point for (F_1, F_2, F_3).

If in addition every pair of elements in X^3 has a lower or an upper bound, then the market equilibrium is unique.

We will consider real-world data sets for oligopoly markets dominated by three large firms. We will attempt to model the response functions of these players using ordered triples of response functions (F_1, F_2, F_3) satisfying the mixed monotone property.

We search for an ordered triple (F_1, F_2, F_3) of linear response functions

$$F_k(x_1, x_2, x_3) = \sum_{j=1}^{3} a_{kj} x_j + d_k, \tag{14}$$

satisfying $\max\left\{\sum_{k=1}^{3} |a_{kj}| : j = 1, 2, 3\right\}$.

We will investigate classes of response functions (F_1, F_2, F_3) with the mixed monotone property. We denote the classes

$$M_i = \{(F_1, F_2, F_3) : a_{kj} \leq 0, a_{ij} \geq 0 \text{ for } i \in \{1,2,3\}\backslash\{k\}, j = 1,2,3\}$$

Let us point out that there is no restriction, which of the players will be put as a first variable in the response functions F_i, which will be the second and so on. Therefore we will consider three different case, mentioned above and besides the mixed monotone property, the contractive condition in Corrolary 1 should be satisfied too.

Models with $(F_1, F_2, F_3) \in M_2$

In our analysis, we have monthly data on beer sales (1000 HLTRs) by five significant corporations that cover the whole Bulgarian beer industry. The percentage share of each company in the market distribution is calculated. To determine the market equilibrium, we consider the three companies with the largest percentage share of sales.

Monthly data on the percentage participation of each company (Com_1, Com_2, Com_3) contains a total of $N = 48$ data from January 2017 until December 2020. Over the period, the three companies covered an average of 88.73%.

Table 1 presents the descriptive statistics on the values of the volume of beer sold for each of the three company brands. The close values of the mean and median, as well as the values close to zero of the skewness and kurtosis for the three variables, indicate that we may assume the data is normally distributed. This is confirmed by the Kolmogorov-Smirnov and Shapiro-Wilk tests, which are insignificant with p-value $> 0,05$ for the variables under consideration. Also, Table 1 shows the third firm has the largest average proportion of sales.

Table 1. Descriptive statistics of the initial variables used.

Variable	**Mean**	**Median**	**Std. Dev.**	**Variance**
Com_1,%	27.593	27.46	2.26	5.09
Com_2,%	25.202	24.95	1.83	3.36
Com_3,%	35.931	35.83	4.18	17.47
Variable	**Skewness**	**Std. Err. Skewness**	**Kurtosis**	**Std. Err. Kurtosis**
Com_1,%	-0.152	0.343	-0.766	0.674
Com_2,%	0.632	0.343	0.102	0.674
Com_3,%	-0.064	0.343	-0.369	0.674

Figure 1 shows a sequence plot of the three companies' percentage participation. The leading company's trend is increasing, while the other two are somewhat decreasing.

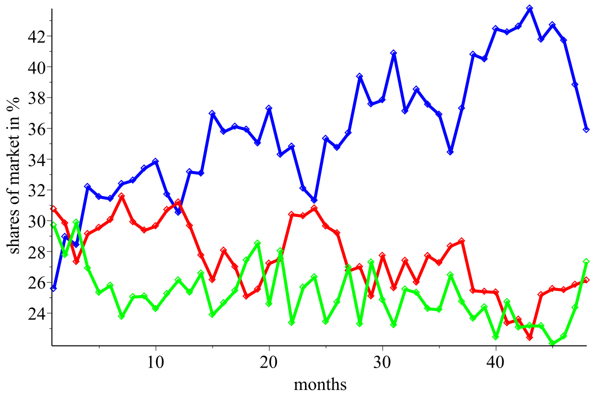
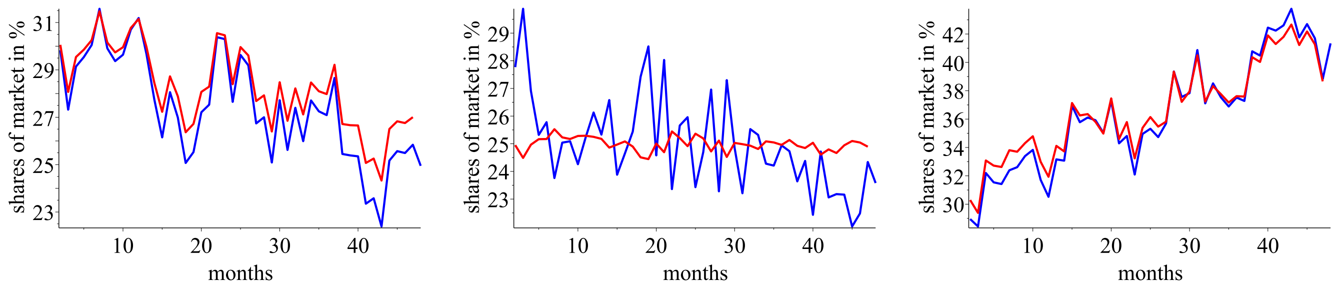


Figure 1. Percentage shares in the market over time for Com_1 in red, Com_2 in green, and Com_3 in blue color.

We get

$$
(15) \qquad \begin{cases} F_1(x,y,z) &= 0.79x & & +0.01z & +6.2, \\ F_2(x,y,z) &= 0.1x & -0.1y & & +24.74, \\ F_3(x,y,z) &= 0.1x & -0.1y & +0.85z & +5.37 \end{cases}
$$



(a) Com_1, real data in blue vs approximated by response function $x_n = F_1(x_{n-1}, y_{n-1}, z_{n-1})$ shares in red

(b) Com_2, real data in blue vs approximated by response function $y_n = F_2(x_{n-1}, y_{n-1}, z_{n-1})$ shares in red

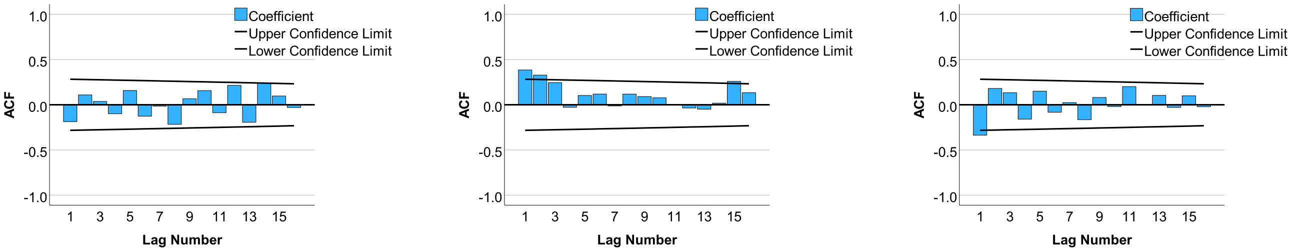
(c) Com_3, real data in blue vs approximated by response function $z_n = F_3(x_{n-1}, y_{n-1}, z_{n-1})$ shares in red

Figure 2. Real data in blue vs approximated by response function in red

Table 2 presents the statistical performance of the created models, which once again shows that the first and third models have high statistical indicators, while the second model explains 25% of the data.

Table 2. Statistical performance of the created models

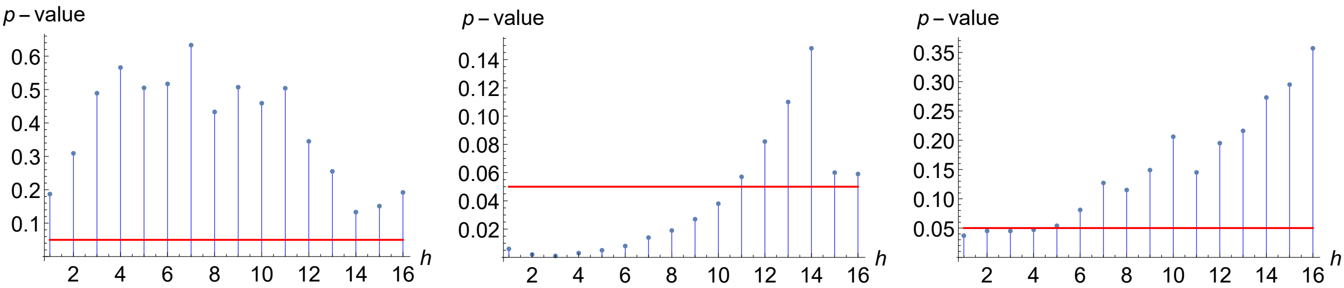
	Model Com_1	**Model** Com_2	**Model** Com_3
R^2	0.988	0.249	0.995
MAPE	0.0302	0.0566	0.0187



(a) Com_1, ACF of reziduals (b) Com_2, ACF of reziduals (c) Com_3, ACF of reziduals

Figure 3. ACF of reziduals of created models.

Figure 3 presents the autocorrelation function (ACF) of the residuals. This indicates that most lags fall within the confidence bounds, suggesting that the residuals behave as white noise. However, a few individual lags exceed the confidence limits, which may be attributed to random fluctuations rather than model misspecification.



(a) Com_1, Ljung-Box Test of reziduals. (b) Com_2, Ljung-Box Test of reziduals. (c) Com_3, Ljung-Box Test of reziduals.

Figure 4. Ljung-Box Test of reziduals of created models.

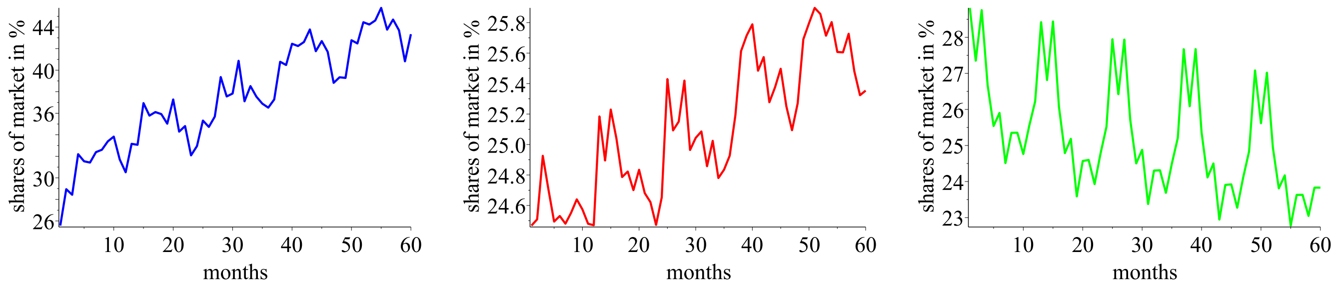
Figure 4 presents the p-values of the Ljung-Box test for the residuals of the constructed models. On the x-axis, the lags h are marked. On the y-axis, the p-values associated with the test for each lag are plotted. A horizontal red line is usually added to indicate the significance level (e.g., 0.05). Lags where the p-values fall below this line indicate the presence of autocorrelation.

In the first model, there are no autocorrelations in the residuals up to the 16th lag. In the third model, a few autocorrelations can be observed up to the second lag, while in the second model, autocorrelations are present up to the 10th lag. This indicates that the first and third models adequately describe the data.

The three models presented, based on response functions satisfying the mixed monotone property, demonstrate the impossibility of describing the market with only one model of class $M_1 \cup M_2 \cup M_3$. It is interesting to note that in each of the considered cases we have a statistically reliable description of two of the market participants. Moreover, if we were to study, for example, the behavior and reactions of two of the participants, for example, the first and second, then if we consider a model with functions of class M_3, we will obtain a model for these two, which predicts their behavior over time and their reactions to changes in the pace. We can conclude that the presented illustrations demonstrate the possibilities of considering three possible models, and in each of them we can draw conclusions for two of the participants.

On illustrative examples with particularly chosen random data, satisfying some patterns

We generate random data that follows a specific pattern. The graphs are presented in the following figure 5. We insist on some periodic behavior of the three players and two of them to show an increasing trend, while the third to be with a decreasing one.

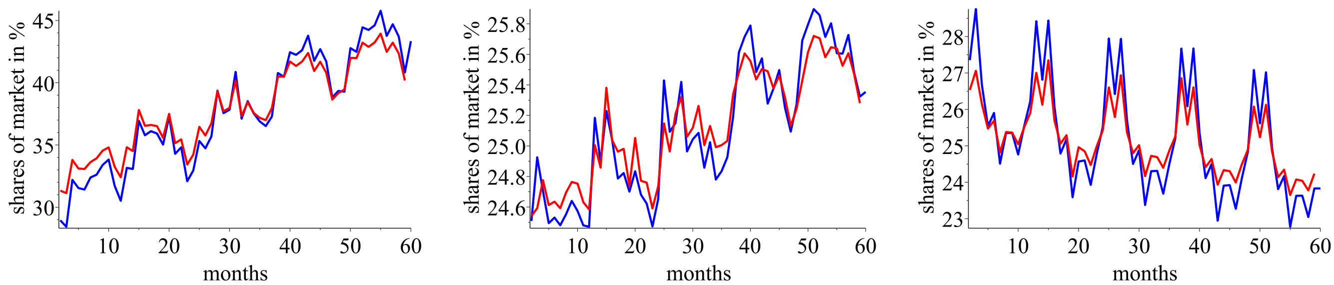


(a) Company 1, real data (b) Company 2, real data (c) Company 3, real data

Figure 5. Random real data for three companies in the oligopoly market

Search for $(F_1, F_2, F_3) \in M_2$, we get

$$
(16) \qquad \left\{
\begin{array}{rclllr}
F_1(x,y,z) & = & 0.79x & & +0.15z & +4.37, \\
F_2(x,y,z) & = & 0.1x & -0.1y & +0.1z & +21.25, \\
F_3(x,y,z) & = & 0.1x & -1.1y & +0.73z & +30.21
\end{array}
\right.
$$



(a) Company 1, real data in blue vs approximated by response function $x_n = F_1(x_{n-1}, y_{n-1}, z_{n-1})$ shares in red

(b) Company 2, real data in blue vs approximated by response function $y_n = F_2(x_{n-1}, y_{n-1}, z_{n-1})$ shares in red

(c) Company 3, real data in blue vs approximated by response function $z_n = F_3(x_{n-1}, y_{n-1}, z_{n-1})$ shares in red

Figure 6. Real data in blue vs approximated by response function in red

We have got a market equilibrium point

$$(x, y, z) = (38.66859470, 25.20445457, 25.05539642).$$

Conclusion

Summary of the obtained results

The main contributions in the present thesis

The main contributions in the present thesis are:

I. Coupled best proximity points results for cyclic and semi-cyclic maps when the underlying space is just a reflexive Banach space, instead of uniformly convex.

II. An error estimation for best proximity points for noncyclic maps has been developed.

III. Coupled fixed points and tripled fixed points for maps with the mixed monotone property in partially ordered metric spaces are investigated.

IV. Ekeland's variational principle for maps with the mixed monotone property is generalized. With the help of it, conditions for the existence and conditions for the uniqueness of tripled fixed points for classes of maps with the mixed monotone property are found.

V. Applications of some of the results in the modeling of oligopoly markets.

List of publications included in the thesis

1 L. Ajeti, A. Ilchev, B. Zlatanov. On Coupled Best Proximity Points in Reflexive Banach Spaces, Mathematics, 10(8), (2022), Article number 1304. (Web of Science, IF=2.258, Q1; SCOPUS, SJR=0.538, Q2)

2 L. Ajeti, B. Zlatanov. Coupled Fixed Points Results for Hardy-Rogers Type of Maps with the Mixed Monotone Property Obtained with The Help of a Variational Technique, MATTEX 2022, CONFERENCE PROCEEDING, v. 1, (2022) 37- 42.

3 L. Ajeti, A. Ilchev. A Variational Principle and Triple Fixed Points, AIP Conference Proceedings, 3182 (2025), Article number 070006, doi: 10.1063/5.0245984, (Web of Science, SCOPUS, SJR=0.152)

Approbation of the obtained results

a) L. Ajeti, B. Zlatanov. Coupled Fixed Points Results for Hardy-Rogers Type of Maps with the Mixed Monotone Property Obtained with The Help of a Variational Technique, MATTEX 2022, CONFERENCE PROCEEDING, v. 1, (2022) 37- 42.

b) L. Ajeti, Coupled best proximity points for noncyclic maps, The International Annual Scientific Conference of the University "Kadri Zeka" (UKZ) Annual Conference 2024, Trends and Challenges in the Development of Contemorary Society, June 13-14, (2024), Gjilan, Republic of Kosovo, pp 27

c) L. Ajeti, A. Ilchev. A Variational Principle and Triple Fixed Points, 49th International Conference - Applications of Mathematics in Engineering and Economics – AMEE'2023, 2023.

The connection between the contributions, the tasks, the paragraphs in the thesis and the included publications.

The connection between the contributions, the tasks, the paragraphs in the thesis and the included publications is as follows:

Contribution	Chapter	Paragraph	Publication	Conference report
I	2	2.1,2.2,2.3	1	
II	2	2.5		b)
III	3	3.1	2	a)
IV	3	3.2	3	c)
V	4	4.3,4.4,4.5	(in progress)	

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