REVIEW

By Assoc. Prof. Miroslav Kolev Hristov PhD.

Shumen University "Bishop Konstantin Preslavski"

Algebra and Geometry Department

of a dissertation for awarding the educational and scientific degree

"Doctor"

in field of higher education 4. *Natural sciences, mathematics and informatics*

professional field: *4.5. Mathematics,*

doctoral program: *Mathematical analysis*

Author: **Laura Ajeti Azemi**.

Topic: „*On Coupled Fixed Points and Coupled Best Proximity Points for Cyclic, Noncyclic, and Semi-Cyclic Maps"*

Scientific supervisors: *Prof.Dr.Sr Boyan Georgiev Zlatanov and Assoc. Prof. Hristina Nikolova Kulina, PhD – University of Plovdiv "Paisii Hilendarski"*

1. General description of the materials presented

By order No. 22-966 /28.04.2025 of the Rector of the University of Plovdiv, I was appointed a member of the scientific jury to ensure a procedure for the defense of the above-mentioned dissertation. At the first meeting of the jury, I was elected a reviewer.

According to Art. 36 (1) of the Regulations for the Development of the Academic Staff of the University of Plovdiv (in force from 16.12.2024), the dissertation candidate has submitted the full set of documents, which includes:

1. a request to the Rector of the PU to disclose the procedure for the defence of a dissertation work;

2. curriculum vitae in European format;

3. protocol from the departmental council for the preliminary discussion of the dissertation work and opinion from the scientific supervisors regarding readiness for preliminary discussion;

4. abstract (in Bulgarian and a foreign language);

4.a. in Bulgarian 64 pages.

4.b. in English 32 pages

4.c. in Bulgarian 32 pages

5. declaration of originality and authenticity of the attached documents;

6. certificate of compliance with the minimum national requirements;

7. list of publications;

8. dissertation work;

9. copies of the publications on the topic of the dissertation work;

10. list of noted citations;

11. document for fee paid, according to the Tariff;

The doctoral student has attached 4 publications.

2. Short biography of the PhD student

Laura Ajeti Azemi was born on 19.04.1993. In 2016, she graduated with a bachelor's degree from the State University of Tetovo - Republic of Macedonia, where she acquired the qualification "teacher of mathematics". A master's degree in mathematics was acquired at the same university in 2020. During the period 2016-2023, Laura Ajeti Azemi worked as a mathematics teacher at Skenderbeu and Ulpiana High Schools. From 2023 to the present, the doctoral student works as an assistant professor at the University of Pristina.

3. Relevance of the topic and appropriateness of the set goals and objectives

In recent years, there has been an avalanche of research in the field of generalizations of various kinds for fixed points and the theory of oligopolistic markets. The interest in fixed points and their generalizations is based on the great opportunities they offer in applications. The interest in oligopolistic markets is dictated by the globalization of the economy. Publications on oligopolistic markets are both purely theoretical in mathematical journals and applied in economics journals. My opinion is that the presented research is relevant and will be interesting for a wide range of readers, both mathematicians and economists.

Also, the first publication of 2022 already has 4 citations that are indexed in the WoS and/or SCOPUS databases, which is convincing evidence of the relevance of the tasks studied in the dissertation..

4. Knowledge of the problem

I believe that at the present time, every researcher is not able to familiarize himself with and assimilate all published results on a given scientific topic (even in a “narrow” direction). The main reasons for this are: the presence of a huge number of sources; externally imposed limited access to information, inaccessible from a linguistic point of view, etc. For the reasons indicated, knowledge of a given scientific problem should mean that the researcher possesses a certain set of scientific information on the topic, which simultaneously has the necessary quality, depth, scope and is sufficient for the specific study. I believe that Laura Ayeti Azemi knows in detail the current state (as well as the historical development) of the scientific problems considered in her dissertation work. This impression of mine is based on the presented dissertation. I will note that the literature contains only sources that are directly related to the research in the dissertation.

5. Research methodology

The main apparatus of conducting the research in the reviewed work are the methods and scientific facts from several mathematical directions: real mathematical analysis; functional analysis; topology; variational principles; theory of oligopolistic markets, statistical methods for modeling on databases. In mathematical research, unlike other sciences, it cannot be said that exactly one method or even several well-known methods are used, but rather different knowledge, skills and techniques are applied, which the researcher manages to combine in order to solve the tasks set.

6. Characteristics and evaluation of the dissertation work

The dissertation presented for defense is 136 pages long, consists of a preface, introduction, three chapters, conclusion and literature of 90 titles.

The introduction is in two parts, a preface where the topic of the researched problems is described as briefly as possible. Chapter 1 is the introduction, where the known concepts and theorems that are used in the following chapters are presented. The concepts of fixed point, partially ordered metric space, uniformly convex and reflexive Banach spaces, pairs and triples of fixed points, points of best approximation, error estimation for fixed points and points of best approximation, Ekeland's variational principle are introduced.

Unlike the classical structure of a dissertation, where the introduction or introduction is dedicated to all the definitions, concepts and results that are of other authors, in the presented dissertation, a good impression is made that the general definition of Cournot market equilibrium for oligopolistic markets and its generalization when using reaction functions instead of profit maximization, statistical methods for evaluating models using real databases are introduced in Chapter 4, where they are used, which makes it easier for the reader. Let us pay attention that the dissertation is in mathematical analysis and Chapters 2 and 3 contain the results in mathematical analysis and are completely sufficient for the acquisition of the educational and scientific degree "doc-

tor". Chapter 4 is additional and illustrates the possibilities of the obtained theoretical results in modeling and researching oligopolistic markets.

Chapter 2 studies Coupled Best Proximity Points for cyclic and semi-cyclic maps. Semi-cyclic maps arise naturally when we consider a market with two participants who change their production levels, taking into account both their own and their competitor's results. On the other hand, the classical Euclidean metric does not describe the profits of the participants well, while the summative one is more suitable for studying oligopolistic markets, with the disadvantage that it does not generate a uniformly convex Banach space, but only a reflexive one. New concepts are introduced, such as Coupled fixed points and Coupled Best Proximity Points for Semi-Cyclic Maps. The notion of a pair of optimal fixed points for noncyclic mappings is generalized to mappings of two variables, sufficient conditions are found for the existence and uniqueness of ordered pairs of optimal points when the adjacent space is uniformly convex, and a priori and a posteriori estimates of the error are found when the modulus of convexity is of power order.

In chapter two, Ekeland's variational principle on sets generated by images with the mixed monotone property in partially ordered metric spaces is generalized. The result thus obtained is used to find sufficient conditions for the existence and uniqueness of triples of fixed points for ordered triples of images with the mixed monotone property. As a consequence, already known results of the direction for triples of fixed points are obtained.Chapter 4 consists of applications in oligopolistic markets, of the results obtained in Chapter 2 and other unpublished results. The unpublished results are on the modeling of real data from an oligopolistic market with three participants. Models of reaction functions for participants that possess the mixed monotonic property are constructed. A statistical analysis of the data predicted by the constructed models is performed. The illustrative examples show that there are markets in which functions with the mixed monotonic property naturally describe the real data statistically reliably.

The conclusion correctly describes the contributions in the presented dissertation work.

The results have been tested at two scientific forums.

7. Contributions and significance of the development for science and practice

The presented dissertation is an original study dedicated to generalizations of Banach's fundamental result on the existence and uniqueness of fixed points. The author's main contributions include:

- Coupled best proximity points results for cyclic and semi-cyclic maps when the underlying space is just a reflexive Banach space, instead of uniformly convexc
- An error estimation for best proximity points for noncyclic maps has been developed.
- Coupled fixed points and tripled fixed points for maps with the mixed monotone property in partially ordered metric spaces are investigated
- Ekeland’s variational principle for maps with the mixed monotone property is generalized. With the help of it, conditions for the existence and conditions for the uniqueness of tripled fixed points for classes of maps with the mixed monotone property are found
- Applications of some of the results in the modeling of oligopoly markets

8. Assessment of publications in the dissertation work

The dissertation work is based on 4 publications (two in journals and two in conference reports). One of the publications is indexed in WoS with Q1 and the other is indexed in SCOPUS with SJR. Two of the works are independent. This shows that Laura Ayeti Azemi meets the minimum national requirements according to the PPZ-RASRB with 105 points out of the required 30.

9. Personal participation of the doctoral student

The doctoral student is trained in an independent form, which implies that upon enrollment as a doctoral student he/she has presented a completed dissertation. The presentation in the dissertation is consistent, technically justified and clearly demonstrates the deep personal engagement of the doctoral student with all stages of the development. The period in which Laura Azemi has been trained and worked on the topic is more than 4 years.

This gives me reason to assume that the formulated contributions and obtained results are her personal merit.

10. The abstract

The abstract correctly systematizes the results of the dissertation and is in accordance with the PU Regulations for the preparation of abstracts for dissertation works.

10. Critical remarks and recommendations

I don’t have critical remarks about the style of the dissertation and the results obtained

I recommend that the author continue his work on the topic of the dissertation, try to solve and publish the open problems set at the end of each chapter and publish the results of chapter 4.

CONCLUTION

In my opinion, the candidate Laura Ajeti Azemi has sufficient, both in quantity and quality, results. The dissertation work contains scientific and applied scientific results that represent an original contribution to science. The submitted documents satisfy all the requirements of the Law on the Development of the Academic Staff in the Republic

of Bulgaria (LAADRB), the Regulations for the Implementation of the LAADRB, the Regulations for the Development of the Academic Staff of the University of Plovdiv. Therefore, I give my strictly positive assessment and recommend that the scientific jury award the educational and scientific degree "doctor" to Laura Ajeti Azemi in the field of higher education 4. Natural Sciences, Mathematics and Informatics, professional field 4.5. Mathematics, doctoral program Mathematical Analysis.

16.05.2025 г.

REVIEWER:.....

(*Assoc. Prof. Miroslav Hristov PhD.*)