REVIEW

by Dr. Vejdi Ismailov Hasanov, professor in
Konstantin Preslavsky University of Shumen

on a dissertation for awarding of the educational and scientific degree “Doctor”
in the field of higher education 4. Natural Sciences, Mathematics and Informatics,
Professional field 4.5 Mathematics
Doctoral program “Mathematical analysis”

Author: Laura Ajeti Azemi

Title: On coupled fixed points and coupled best proximity points for cyclic, noncyclic, and semi-cyclic maps

Scientific supervisors:

Prof. DSc Boyan G. Zlatanov – University of Plovdiv “Paisii Hilendarski”
Assoc. prof. Dr. Hristina N. Kulina – University of Plovdiv “Paisii Hilendarski”

1. General description of the presented materials

By Order No. PD-22-966 / 28.04.2025 of the Rector of University of Plovdiv "Paisiy Hilendarski" (PU), I have been appointed as a member of the scientific jury to ensure a procedure for the defense of a dissertation on the topic "On pairs of fixed points and pairs of points of best approximation for cyclic, non-cyclic and semi-cyclic images" for the award of the educational and scientific degree "doctor" in the field of higher education 4. Natural sciences, mathematics and informatics, professional field 4.5. Mathematics, doctoral program "Mathematical Analysis". The author of the dissertation is Laura Aieti Azemi – a PhD student in independent preparation at the Department of "Mathematical Analysis" with scientific supervisor Prof. DSc Boyan Georgiev Zlatanov and Assoc. Prof. Dr. Hristina Nikolova Kulina from PU.

The set of materials presented by Ms. Laura Azemi is in accordance with Art. 36 (1) of the Regulations for the Development of the Academic Staff of the University of Plovdiv, and includes the following documents:

- a request to the Rector of the University of Plovdiv to open a procedure for the defence of a dissertation work;
- a curriculum vitae in European format;

– a statement of the scientific supervisors on the readiness of the dissertation for preliminary discussion and a report from the department council on this discussion and a decision on the suitability of the work for defence before a scientific jury;
– dissertation work (in English);
– an abstract (in Bulgarian and English);
– a list of scientific publications on the subject of the dissertation;
– copies of scientific publications;
– a list of noted citations;
– a declaration of originality and authenticity of the attached documents;
– a certificate of fulfillment of the minimal national requirements.

The PhD student has submitted a list of 4 publications and copies of them. One of the copies of publications is in fact an abstract, which I accept as proof of participation in a conference.

2. Brief biographical data

According to the submitted CV, Ms. Laura Azemi completed her higher education in 2016 at the State University of Tetovo, Tetovo, North Macedonia, obtaining a Bachelor's degree with the qualification "Teacher of Mathematics". In 2020, she obtained a Master's degree at the same university with the qualification "Mathematician".

Ms. Laura Azemi's professional career began in 2016 as a mathematics teacher at Skenderbeu High School, Presevo, Serbia. Later that year, she joined Ulpiana High School, Lipjan, Kosovo (until November 2023). Since October 17, 2023, she has been an assistant professor of mathematics at the University of Belgrade, Pristina, Kosovo.

3. Relevance of the topic and appropriateness of the set goals and tasks

The subject of the dissertation work is on generalizations of Banach's fixed point theorem (1922) for contractive maps related to coupled fixed points and coupled best proximity points, as well as on their possible applications.

Fixed point theorems of a map are an important tool for proving the existence of a solution to problems in various scientific fields. From the fundamental theorems: of Brouwer (1911), its generalizations – the theorem of Schauder (1930), Kakutani (1941); of Banach, which was mentioned, followed by the theorem of Brouwer (1965), of Ran and Reurings (2004), etc., until now, research on this topic has not lost its relevance. Research aimed at generalization of the Banach's theorem is mainly in two directions: the first is changes in the requirements on the underlying space, and the second – in the requirements for the maps under consideration. In recent years,

interest in coupled and tripled fixed points and coupled and tripled best proximity points of maps with mixed monotonicity property, defined on partially ordered spaces, has increased.

The aim of the research in the dissertation work is to obtain new results on coupled fixed points and coupled best proximity points under weakened conditions on the underlying space and secondly on noncyclic and semi-cyclic maps. Along with research on coupled fixed points, the problem of tripled fixed points of maps with mixed monotonicity property in partially ordered spaces is considered.

The relevance of the research topic and the expediency of the problem posed is indicated by numerous publications by other authors on the topic and the considered example of an application for determining market equilibrium for oligopoly markets.

4. Knowledge of the problem and research methodology

Ms. Laura Azemi has included 90 sources in the bibliography of her dissertation work. Each source is selected according to the topic of the research and shows the development of the theory of coupled fixed point and coupled best proximity points. All sources except one are in English, with over 30 being from the last 15 years. The review made, the preliminary results presented in the introduction of the dissertation and the correct citation of the sources are respectively a sign that Ms. Azemi is well aware of the state of the problem and appreciates the achieved preliminary results and the possibilities for their development.

The methodology and research methods used are mainly theoretical. The conducted empirical study of the state of the problem, followed by analysis of the state and evaluation of the results are a prerequisite for achieving the set goals and objectives. By the method of analogy and generalization of results for pairs of points, results for three points and corresponding applications are obtained.

5. Characteristics and evaluation of the dissertation work

The dissertation work is 136 pages and includes four chapters, a conclusion and a bibliography.

The first chapter of the dissertation is an introduction to the subject and is of a descriptive nature. The state of the problem is presented in detail, the research tasks are formulated and the developed topic is justified. The main terms, definitions and statements from the literature are given. The presentation is detailed and comprehensive and provides clarity about the significance and relevance of the research topic.

The second chapter is devoted to couple best proximity points for pairs of cyclic, semi-cyclic and noncyclic maps. The content of the chapter is based on publication [1] from the attached list of publications. From the reference it is evident that the results were presented in a report [4] at a scientific conference. In paragraph 2.1, studies were carried out for coupled best proximity points (ξ_x, ξ_y), (η_x, η_y) of a cyclic contraction ordered pair of ordered pairs maps $((F, f), (G, g))$ in a reflexive Banach space. In the following paragraphs, pairs of semi-cyclic and noncyclic contraction maps in a reflexive and uniformly convex Banach space, respectively, are studied.

In the third chapter, the Hardy-Rogers map of two variables with mixed monotonicity in a partially ordered space is investigated and the application of Ekeland's variational principle for tripled fixed points to the map with mixed monotonicity property is investigated. The results in the third chapter are published in [2,3].

In the fourth chapter, the application of fixed points and best approximation points in modelling market equilibrium in oligopoly markets is examined. Duopoly markets and markets dominated by three players are considered.

6. Contributions and significance of the development for science and practice

The contributions in this dissertation are of a scientific and applied nature.

The existence of coupled best proximity points (ξ_x, ξ_y) (η_x, η_y) to (F, f) and (G, g) respectively is proved, as $((F, f), (G, g))$ is a cyclic contraction ordered pair of pairs of maps that are defined in subsets of a reflexive Banach space *X*. The relaxation of the requirement for *X* from uniformly convex to reflexive Banach space is at the expense of the uniqueness of the coupled best proximity points. For a semi-cyclic pair of contraction maps (F, G)in a reflexive Banach space, the existence of a couple best proximity point (ξ, η) is also proved. It is proven that for a noncyclic contraction pair of maps (F, G) of two variables there exists a unique optimal pair $((x, y), (u, v))$ of ordered coupled fixed points (x, y) and (u, v) of *F* and *G*, respectively. In this case, an estimate of the errors $\|x - x_n\|$ and $\|y - y_n\|$ is obtained.

For a Hardy–Roger's map of two variables with mixed monotone property in a partially ordered space, the existence of a coupled fixed point is proven. For an ordered triple of maps (F_1, F_2, F_3) with mixed monotone property, the Ekeland principle is applied and as an application, the existence of a tripled fixed point (x, y, z) of (F_1, F_2, F_3) is proven.

7. Evaluation of dissertation publications

The main results of the dissertation work have been published in three publications. All three publications are co-authored, two of which have one co-author and one with two co-authors. In

two of the works, one co-author is the same scientific supervisor. I believe that the contributions of the co-authors in the publications are equivalent.

Paper [1] from the presented list of publications contains the main results of the second chapter and was published in the journal Mathematics with an impact factor in quartile Q1 of the Web of Science. Publication [2] in the proceedings of the scientific conference MATTECH 2022 with international participation from the national reference list. The third publication [3] is included in a proceeding of the 49th International conference "Application of mathematics in technology and economics" 2023, edition of AIP Conference proceedings, which is with SJR and is referenced in Scopus and Web of Science. The two publications [2,3] include results from the third chapter and are the basis for the applications in the fourth chapter.

The results have been tested at a total of three scientific conferences. A search in Scopus shows that publication [1] has 4 citations. According to the minimal national requirements in the Regulations for the Implementation of the Law on the Development of the Academic Staff in the Republic of Bulgaria, the necessary and accumulated points per group of indicators are as follows:

Indicator group	Minimal points	Accumulated points
A – dissertation work	50	**50 points.**
Г 7 – scientific publications - in the IF edition of Q1 of WoS - in edition with SJR	30	**Total 105 points** 3*25=75 3*10=30
Д 11 – citations	-	**4*2*4=32 points**

Therefore, the minimal national requirements are exceeded.

I have not noticed any plagiarism, both in the publications and in the dissertation work of Ms. Laura Azemi.

8. Personal participation of the doctoral student

I have no reason to doubt the doctoral student's personal participation in the dissertation research, her contributions, and her preparation of the dissertation work.

9. Abstract

The abstract is prepared in two languages – Bulgarian and English. It correctly reflects the content of the dissertation work and meets the regulatory requirements. It contains some technical and linguistic errors, but they do not belittle the achieved results. There is a discrepancy in the

numbering of the chapters in the abstract and those in the dissertation work, due to the lack of a number for the introduction in the abstract, while in the dissertation work it has the number one.

10. Critical remarks and recommendations

I do not have critical remarks on the set of documents.

The dissertation work contains some technical errors, which can be easily removed with a more careful reading. I recommend considering Definition 1.9 and the mixed monotone property of a pair of maps. I believe that here $G(x_1, y) \preccurlyeq G(x_2, y)$ should be $G(x_1, y) \succcurlyeq G(x_2, y)$. Definition 1.41 contains two defined sequences, which could be as numbered formulas to be cited if necessary. The text "Everywhere, when considering the sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ we will assume that they are the sequences defined in Definition 1.41" is not applicable to the entire work (see Theorem 2.3). I recommend that all used notations be defined in the conditions of the theorems or that a reference be made so that their meaning is clear. I draw attention to Theorem 2.3, in addition to the sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$, and $\{v_n\}_{n=0}^{\infty}$, the constants C, q and α, which are related to the convexity of the space *Z* and the property of the pair of maps (*F*, *G*), are not explained.

The following are some of the inaccuracies noted in the dissertation:

- on page 8, line 19, $X \times X$ should be $X \times Y$;
- on page 11, in Theorem 1.4, $(X \times X, \rho)$ is unnecessary;
- on page 13, in Theorems 1.5 and 1.6 and on page 4, lines 4 and 6, f should be *T*;
- on page 18, in Theorem 1.9, $y = F(\eta, \xi, \eta)$ should be $\eta = F(\eta, \xi, \eta)$;
- on page 25, in Theorem 1.17, $(X, \|\cdot\|^*)$ should be $(X^*, \|\cdot\|^*)$;
- on page 68, in line 6 from bottom to top, *A* and *B* should be *X*;
- on page 70, in line 12, $u_n = F(u_n, v_n)$ and $v_n = F(v_n, u_n)$ should be $u_n = G(u_n, v_n)$ and $v_n = G(v_n, u_n)$;
- on page 79, in line 15 from bottom, $x \preccurlyeq u$ and $z \preccurlyeq w$ should be $u \preccurlyeq x$ and $w \preccurlyeq z$, in Definition 3.11, $H: X \to X$ should be $H: X \times X \times X \to X$, in Definition 3.12, "of if" should be "of (*F*,*G*,*H*) if";
- on page 80, at the end of Proposition 3.1, the inequalities$\succcurlyeq x$, $v \preccurlyeq y$ and $w \succcurlyeq z$ are r unnecessary, they are satisfied by condition.

11. Personal impressions

I don't know the doctoral student and I have no prior impressions.

CONCLUSION

The dissertation contains scientific and scientific-applied results that represent an original contribution to science and meet all the requirements of the Law on the Development of the Academic Staff in the Republic of Bulgaria (LDASRB), the Regulations for the Implementation of the LDASRB and the relevant Regulations of PU. The content of the dissertation work and the publications show that the doctoral student Laura Ajeti Azemi has in-depth theoretical knowledge and professional skills in the professional field 4.5. Mathematics, demonstrating qualities and skills for independent conduct of scientific research.

Due to the above, I give my positive assessment for the conducted research, the achieved results and contributions in the presented dissertation work and I propose to the honorable scientific jury to award the educational and scientific degree "doctor" to Laura Ajeti Azemi in the field of higher education: 4. Natural Sciences, Mathematics and Informatics, professional field 4.5. Mathematics, doctoral program "Mathematical Analysis".

16.05.2025 Reviewer: .....

(Prof. Dr. Vejdi Hasanov)