REVIEW

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of a dissertation for awarding the educational and scientific degree "doctor"
in the field of higher education 4. Natural sciences., mathematics and informatics
professional direction 4.5 Mathematics
PhD program Mathematical analysis

Author: Mira Lachezarova Spasova
Topic: Analytical methods for solving some classes of fuzzy integro-differential equations
Research supervisor: Professor Atanaska Tencheva Georgieva, Ph.D.,
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1. General description of the presented materials

By order No. RD 21-454 dated 23.02.2024 of the Rector of the Plovdiv University "Paisii Hilendarski" (PU), I have been appointed as a member of the scientific jury to ensure a procedure for the defense of a dissertation work on the topic "Analytical methods for solving some classes of fuzzy integro-differential equations" for obtaining the educational and scientific degree 'doctor' in the field of higher education 4. Natural sciences, mathematics and informatics, professional direction 4.5. Mathematics, doctoral program Mathematical analysis.

The author of the dissertation is Mira Lachezarova Spasova - a full-time doctoral student at the Department of "Mathematical Analysis" with scientific supervisor Professor Atanaska Tencheva Georgieva, Ph.D., University of Plovdiv "Paisii Hilendarski".

The set of paper materials presented by Mira Lachezarova Spasova is in accordance with Art. 36 (1) of the Regulations for the Development of the Academic Staff of the PU. The doctor has attached 4 publications.

2. Brief biographical data about the doctoral student

Mira Lachezarova Spasova was born on 19.02.1995. She graduated with a bachelor's degree "Mathematics, Informatics and Information Technology Teacher" at the "Paisii Hilendarski" PU in 2018 and a master's professional qualification "Mathematics" in 2019, again in PU "Paisii Hilendarski". After that, she worked as a teacher in various high schools.

Mira Spasova is enrolled as a full-time doctoral student in the field of higher education 4. Natural sciences, mathematics and informatics, professional direction 4.5. Mathematician, PhD program Differential Equations at the "Mathematical Analysis" department of the Faculty of Mathematics and Informatics (FMI) at Plovdiv University "Paisii Hilendarski" with a duration of 3 years. Prof. Atanaska Tencheva Georgieva, Ph.D., has been appointed as the scientific supervisor and the topic of the dissertation is Analytical methods for solving some classes of fuzzy integro-differential equations.

By order of the rector, she was dismissed with the right of defense, starting from 01.03.2024.

3. Actuality of the topic and appropriateness of the set goals and problems

The dissertation consists of 107 pages. The cited literature contains 103 titles of articles and monographs. The PhD student has presented research on Volterra-Fredholm fuzzy integro-differential equations, Volterra fuzzy integro-differential equations and Volterra fuzzy partial integro-differential equations. With the application of various analytical methods, approximate and exact solutions of the considered equations were found. New results were obtained and published in the Conference Proceedings of the American Institute of Physics. They have also been reported at five international conferences.

The results obtained in the dissertation work are actual. They have been published in four articles co-authored with her supervisor.

4. Knowing the problem

I believe that the doctoral student is well acquainted with the current state and historical development of the examined scientific problems. The basis for the statement made are the following facts:

- The author's comprehensive historical overview in the introduction to the dissertation work;
- Content-rich introduction to the dissertation topic. Basic definitions and results of functional analysis and the theory of fuzzy differential and integro-differential equations are presented;
- The literature, which includes 103 titles, includes the main results of leading authors in the given scientific field.

5. Research methodology

The author uses the traditional methods of real and functional analysis and methods for the study of differential equations, modified and generalized for fuzzy differential and integro-differential equations.

6. Characterization and evaluation of the dissertation work

The present dissertation is dedicated to finding approximate and exact solutions of some classes of fuzzy integro-differential equations, using analytical methods.

Chapter one is an overview and in it are given basic definitions and theorems that are used in the dissertation work. It consists of 5 paragraphs.

The main contributions of the thesis are contained in Chapters 2, 3 and 4, therefore I will briefly dwell on the results obtained there.

The second chapter consists of 3 paragraphs, in which there are separate sub-paragraphs, for greater clarity of the study.

In §2.1, the Adomian decomposition method for the nonlinear fuzzy Volterra Fredholm integro-differential equation is discussed. In Subsection §2.1.1, a statement of the problem for this equation is made. In Subsection § 2.1.2 the parametric form of the equation is given. In Subsection

§2.1.3, a fuzzy version of the Adomian decomposition method is constructed and applied to find the approximate solution of the studied equation.

Sufficient conditions for the existence and uniqueness of the solution of the equation are found in Subsection §2.1.4. The main result is: *Theorem 2.1.1*. Let the conditions (i) − (iii) be satisfied. Then the integral equations (3) and (4) have a unique solution.

In Subsection § 2.1.5, the convergence of the method is proven and an estimate of the error between the exact and the approximate solution of the studied equation is obtained. The other main result is *Theorem 2.1.3*. Let conditions (i)−(iii) hold. Then the maximum absolute error of the solution (5) of the integral equations (3) and (4) is given by the inequalities (9), (10).

In § 2.2, the fuzzy Sumudu transform (FST) is defined. Some of its properties are given and it is applied to fuzzy derivatives. Subsection § 2.2.1 gives a definition of FST and its inverse. In Subsection § 2.2.3 fuzzy convolution is presented. In Subsection § 2.2.4, basic properties of FST related to fuzzy derivatives are given.

In § 2.3, a fuzzy Sumudu decomposition method (FSDM) is constructed, which is a combination of the fuzzy Sumudu transform and the fuzzy Adomian decomposition method. In Subsection § 2.3.1, a statement of the task for FSDM is made. In Subsection § 2.3.2, the fuzzy Sumudu transform for equation (16) is applied.

The third chapter consists of 3 paragraphs, in which there are also separate sub-paragraphs, for greater clarity of the study.

In §3.1, the formulation of the problem for the linear fuzzy Volterra integro-differential equation (LFVIDE) is made.

In §3.2, the Natural transform of the Fourier integral is derived and related to the Laplace and Sumudu transforms. In Subsection § 3.2.1, a definition of the fuzzy Natural transform (FTN) and the relationship between them is given. In Subsection § 3.2.2 properties of PTH are given. In Subsection § 3.2.3 the fuzzy convolution is given. In Subsection § 3.2.4, new results related to FTNs for fuzzy derivatives of the m-th order are obtained.

In §3.3, Natural's fuzzy transform is applied to study Volterra's fuzzy linear integro-differential equation.

The fourth chapter consists of 3 paragraphs.

In §4.1, the formulation of the problem for the linear fuzzy Volterra partial integro-differential equation (FLIDE) is made.

In §4.2, the fuzzy Sumudu transform is used to solve fuzzy partial integro-differential equations. Subsection § 4.2.1 gives a definition of the FST for a function of two variables and its inverse. In addition, some of its main properties have been proven. In Subsection § 4.2.2, a definition and theorem for fuzzy convolution are given. In Subsection § 4.2.3, the main properties of the FST for a function of two variables associated with partial fuzzy derivatives are obtained. In Subsection §4.2.3, the FST method is used for the investigated equation, which is reduced to a fuzzy ordinary differential equation.

In §4.3, Natural's fuzzy transform for solving fuzzy partial integro-differential equations is studied. Subsection § 4.3.1 gives a definition of a fuzzy two-dimensional Natural transform (FDTN) for a function of two variables and its inverse. In Subparagraph § 4.3.2 basic properties of FDTN are given. In Subsection § 4.3.3, a definition and theorem for fuzzy convolution are given. In Subsection § 4.3.4, new results for FDTN for fuzzy partial derivatives of the m-th order are obtained. In Subparagraph § 4.3.5, FDTN is used to find the exact solution of the LFVIDE.

7. Contributions and significance of the development for science and practice

Basically, I accept the doctoral student's claims for the contributions in the dissertation, which I formulate as follows:

1. Sufficient conditions for the existence-uniqueness of the solution of a nonlinear fuzzy Voltera-Fredholm integro-differential equation are found.
2. A fuzzy analytical method is constructed using the Adomian decomposition method to find an approximate solution of a nonlinear fuzzy Volterra-Fredholm integro-differential equation. Sufficient conditions for the convergence of the method are found and an error estimate is obtained. A fuzzy transform of Sumudu is constructed. Sufficient conditions for the existence of the transformation and its application to ordinary and partial fuzzy derivatives are found.
3. A fuzzy analytical method is constructed which is a combination of Sumudu fuzzy transform and Adomian decomposition to find the approximate solution of nonlinear fuzzy Volterra-Fredholm integro-differential equation.
4. A fuzzy Natural transform is constructed to find an exact solution of a linear fuzzy Volterra integro-differential equation with a convolutional kernel. Sufficient conditions for the existence of the transformation and its connection with the Laplace and Sumudu transformations have been found. A fuzzy analytical method is also proposed, which uses the fuzzy variant of the Sumudu transform to find the exact solution of a linear partial fuzzy Volterra integro-differential equation.
5. A fuzzy two-dimensional Natural transform is constructed to find the exact solution of a linear partial fuzzy Volterra integro-differential equation. We have found sufficient conditions for the existence of the transformation and its application for fuzzy partial derivatives.

8. Critical remarks and recommendations

I had some questions that we clarified in conversations with the PhD student. She responded to me in a timely manner, so I won't comment on the details.

9. Evaluation of publications on the dissertation work

There are 4 articles of the doctoral student related to the dissertation. They were published in an authoritative journal. The presented table shows the points that are necessary for the requirements of the Law on the Development of the Academic Staff in the Republic of Bulgaria (ZRASRB). Some of the results were reported at five conferences. There is also participation in several projects.

10. Personal participation of the doctoral student

The authorship of the doctoral student in the presented works is indisputable, as they are written in the same manner of exposition. Contribution in co-authored publications is appreciable and leaves no room for doubt. I have not found any elements of plagiarism.

11. Abstract

The abstract contains 33 pages and correctly reflects the results obtained, highlighting the main contributions. All statements are made without evidence.

12. Personal impressions

I have no personal impressions.

13. Recommendations for future use of dissertation contributions and results

I recommend Mira Spasova to continue to deal with the researches she has started.

Conclusion

The dissertation contains new scientific results that meet all the requirements of the ZRASRB, the Regulations for the implementation of the ZRASRB and the relevant Regulations of the PU "Paisii Hilendarski". The presented materials and dissertation results fully correspond to the specific requirements of the Faculty of Mathematics and Informatics, adopted in connection with the Regulations of the PU for the application of ZRASRB.

The dissertation shows that the doctoral student Mira Lachezarova Spasova possesses in-depth theoretical knowledge and professional skills in the scientific specialty Mathematical analysis, demonstrating qualities and skills for independent conduct of scientific research.

In conclusion, I give my **positive assessment** of the conducted research, presented by the above-reviewed dissertation work, abstract, achieved results and contributions, and I propose to the honorable scientific jury to award the educational and scientific degree "doctor" to Mira Lachezarova Spasova in the field of higher education: 4. Natural sciences, mathematics and informatics, professional direction 4.5 Mathematics, doctoral program Mathematical analysis.

04.04.2024 /prof. d.n. Vasil Angelov/