”Paisii Hilendarski” University of Plovdiv
Faculty of Mathematics and Informatics

Department of ”Mathematical Analysis”

Mira Lachezarova Spasova

Analytical methods for solving some classes of fuzzy integro-differential equations

ABSTRACT

of a dissertation
for awarding the educational and scientific degree
”DOCTOR”
field of higher education:
4. Natural sciences, mathematics and informatics;
professional direction: 4.5. Mathematics;
PhD program: Mathematical Analysis

Supervisor:
Prof. Dr. Atanaska Tencheva Georgieva

Plovdiv – 2024

"Paisii Hilendarski" University of Plovdiv
Faculty of Mathematics and Informatics

Department of "Mathematical Analysis"

Mira Lachezarova Spasova

Analytical methods for solving some classes of fuzzy integro-differential equations

ABSTRACT

of a dissertation

for awarding the educational and scientific degree

"DOCTOR"

field of higher education:

4. Natural sciences, mathematics and informatics;

professional direction: 4.5. Mathematics;

PhD program: Mathematical Analysis

Supervisor:
Prof. Dr. Atanaska Tencheva Georgieva

Plovdiv – 2024

The dissertation work has been discussed and scheduled for the defense of an extended departmental council of the Department of "Mathematical Analysis" at the Faculty of Mathematics and Informatics of Plovdiv University "Paisiy Hilendarski", held on 12.02.2024.

The dissertation work "Analytical methods for solving some classes of fuzzy integro-differential equations" consists of an introduction, four chapters, a conclusion and a bibliography. The bibliography contains 103 titles. The total volume of the dissertation is 107 pages. The list of author publications includes 5 titles.

The defense materials are available for those interested in the FMI secretariat, New Building of PU "Paisiy Hilendarski", 236 "Bulgaria" Blvd., cab. 330, every working day from 8:30 AM to 5:00 PM.

The numbering of theorems, lemmas, remarks, and definitions in the abstract includes your numbering in the dissertation.

Contents

Introduction

Actuality of the dissertation

One of the main tools of applied mathematics is the integral equation. Many branches of science and engineering naturally contain integral equations. Functional equations such as partial differential equations, integral and integro-differential equations, stochastic equations, and others are often created when real-world problems are modeled mathematically. Integro-differential equations are the general component of mathematical descriptions of physical phenomena. They can be found in fluid dynamics, biological models, and chemical kinetics. Integra-differential equations arise in numerous physical processes, including the formation of glass [31], nanohydrodynamics [20], droplet condensation [28], wind waves in the desert [7] and biological model [25].

In some cases, information about real-life problems encountered is fraught with uncertainty. This uncertainty results from several factors, such as measurement errors, insufficient data, or if limiting conditions are introduced. So it is necessary to have mathematical tools to understand this uncertainty. Therefore, the formation of a convenient and applicable algorithm is important to achieve an accurate mathematical structure to process and solve them.

Fuzzy differential and integral equations are a powerful tool for modeling dynamical systems describing processes and phenomena from mathematical physics, fuzzy financial and economic systems, and fuzzy financial mathematics. They are characterized by data that is not precisely defined [34, 11, 27, 32, 35] there is a loss of some of them or they are obtained from more than one source.

In recent years, many scholars have contributed to the research and study of the solutions of fuzzy integro-differential equations using various numerical and analytical techniques. These techniques include the homotopy perturbed method [1, 24], Picard method [5, 26] Laplace and Adomian decomposition method [2, 9, 16], Sumudu decomposition method [3, 21], fuzzy differential - transform method [6, 23], generalized linear method [22] and other. The existence and uniqueness of the solution of fuzzy integro-differential equations are studied in [12, 19, 26, 30].

The fuzzy theory of partial differential and integro-differential equations is a new and important branch of fuzzy mathematics. It has wide applications due to the fact that many practical problems in industrial engineering, computer science, physics, artificial intelligence, and operations research can be converted to imprecise partial values. The topic of fuzzy partial integro-differential equations has attracted the attention of researchers recently because it is considered a powerful tool to represent fuzzy parameters and deal with their dynamical systems in natural fuzzy environments. Indeed, it has great significance in the theory of fuzzy analysis and its applications in fuzzy control models, artificial intelligence, quantum optics, atmospheric measurement theory, etc. [4, 8, 10, 12].

Purposes and objectives of the dissertation work

The main objects of research in the dissertation work are nonlinear fuzzy Volterra-Fredholm integro-differential equation, linear fuzzy Volterra integro-differential equation and linear partial fuzzy Volterra integro-differential equation.

The main objectives of the dissertation work are the following:

1. To expand the mathematical apparatus of nonlinear fuzzy integro-differential equations, necessary to study the existence and uniqueness of their solution.

2. To construct fuzzy decomposition methods for finding the approximate solutions of the nonlinear fuzzy Volterra-Fredholm integro-differential equation.

3. To define and investigate fuzzy integral transformations necessary to find the exact solutions of a linear fuzzy Volterra integro-differential equation and a linear partial fuzzy Volterra integro-differential equation.

The objectives of this dissertation work have been achieved by solving the following tasks:

a) Finding sufficient conditions for the existence and uniqueness of the solution of a nonlinear fuzzy Volterra-Fredholm integro-differential equation.

b) Construction of a fuzzy variant of the Adomian decomposition method to find the approximate solution of a nonlinear fuzzy Volterra-Fredholm integro-differential equation. Finding sufficient conditions for the convergence of the method and obtaining an estimate of the error.

c) Defining and study of fuzzy variant of Sumudu transform. Its use to construct a fuzzy decomposition method to find the approximate solution of the nonlinear fuzzy Volterra-Fredholm integro-differential equation.

d) Defining and study of fuzzy transformation of Natural. Its application to find the exact solution of a linear fuzzy Volterra integro-differential equation, with fuzzy convolution.

e) Using the Sumudu fuzzy transform to find the exact solution of Volterra's linear partial fuzzy integro-differential equation.

f) Defining and study of the fuzzy two-dimensional Natural transform. Its application to find the exact solution of Volterra's linear partial fuzzy integro-differential equation.

Structure of the dissertation

The present dissertation work is dedicated to finding approximate and exact solutions of some classes of fuzzy integro-differential equations, using analytical methods. It contains 107 pages and consists of an introduction, four chapters, a conclusion and a bibliography. Contains 9 graphs and 1 table.

Brief overview of the dissertation work

Chapter 1. A brief overview

Chapter one is overview and it gives basic definitions and theorems that are used in the dissertation work. It consists of 5 paragraphs.

In **Section § 1.1** the essence of fuzzy sets, their definition and operations on fuzzy sets are given.

In **Section § 1.2** gives a definition of a fuzzy number as well as some of its representations, the arithmetic of fuzzy numbers and the Hukuhara difference.

In **Section § 1.3** gives definitions and basic properties for fuzzy functions of one variable, fuzzy derivative, and integral.

In **Section § 1.4** gives definitions and basic properties for fuzzy functions of two variables, fuzzy partial derivative and fuzzy integral.

In **Section § 1.5** discusses Volterra's population model, which describes population growth within a closed system.

Chapter 2. Decomposition methods for solving NFIDE

Chapter 2 consists of 3 paragraphs in which there are separate sections for greater clarity of the study.

In **Section § 2.1** discusses the Adomian decomposition method for the nonlinear fuzzy Volterra-Fredholm integro-differential equation (NFIDE).

In **Subsection § 2.1.1** a statement of the task is made for NFIDE.

$$\begin{gathered}\sum_{j=0}^{k} p_j \odot u^{(j)}(x) = g(x) \oplus (FR) \int_a^x k_1(x,s) \odot G_1(u(s))ds \oplus \\ \oplus (FR) \int_a^b k_2(x,s) \odot G_2(u(s))ds,\end{gathered} \tag{1}$$

with initial conditions

$$u^{(j)}(a) = b_j, \quad j = 0, 1, 2, ..., k-1, \tag{2}$$

where $p_j : [a,b] \to \mathbb{R}$, k_1, $k_2 : [a,b] \times [a,b] \to \mathbb{R}$, G_1, $G_2 : E^1 \to E^1$ are continuous functions in E^1 and g, $u : [a,b] \to E^1$ and are continuous fuzzy numerical values of the functions $b_j \in E^1$, $j = 0, 1, ..., k-1$ and $a, b \in \mathbb{R}$.

In **Subsection § 2.1.2** is given the parametric form of equation

1.

$$\begin{aligned}
\underline{u}(x,r) &= \tfrac{1}{p_k} L^{-1}(\underline{g}(x,r)) + \tfrac{1}{p_k k!} \int\limits_a^x (x-s)^k k_1(x,s) G_1(\underline{u}(s,r))ds + \\
&+ \tfrac{1}{p_k} \int\limits_a^b L^{-1}(f_2(x)) h_2(s) G_2(\underline{u}(s,r)))ds - \\
&- \tfrac{1}{p_k (k-1)!} \sum_{j=0}^{k-1} p_j \int\limits_a^x (x-s)^{(k-1)} \underline{u}^{(j)}(s,r)ds + \\
&+ \sum_{j=0}^{k-1} \tfrac{1}{j!} (x-a)^j \underline{b}_j(r),
\end{aligned} \tag{3}$$

Similarly we obtain

$$\begin{aligned}
\overline{u}(x,r) &= \tfrac{1}{p_k} L^{-1}(\underline{g}(x,r)) + \tfrac{1}{p_k k!} \int\limits_a^x (x-s)^k k_1(x,s) G_1(\underline{u}(s,r))ds + \\
&+ \tfrac{1}{p_k} \int\limits_a^b L^{-1}(f_2(x)) h_2(s) G_2(\underline{u}(s,r)))ds - \\
&- \tfrac{1}{p_k (k-1)!} \sum_{j=0}^{k-1} p_j \int\limits_a^x (x-s)^{(k-1)} \underline{u}^{(j)}(s,r)ds + \\
&+ \sum_{j=0}^{k-1} \tfrac{1}{j!} (x-a)^j \underline{b}_j(r),
\end{aligned} \tag{4}$$

In **Subsection § 2.1.3** a fuzzy variant of the Adomian decomposition method is constructed and applied to find the approximate solution of the studied equation.

Let the unknown functions $(\underline{u}(x,r), \overline{u}(x,r))$ are of the kind

$$\underline{u}(x,r) = \sum_{i=0}^{\infty} \underline{u_i}(x,r), \quad \overline{u}(x,r) = \sum_{i=0}^{\infty} \overline{u_i}(x,r) \tag{5}$$

Denote $$D^j(u(x)) = \frac{d^j u(x)}{dx^j}, \quad j = 0, 1, ..., k-1.$$

The nonlinear operators $G_1(\underline{u})$, $G_1(\overline{u})$, $G_2(\underline{u})$, $G_2(\overline{u})$, $D^j(\underline{u})$ and

$D^j(\overline{u})$ are infinite series of polynomials defined by the equalities

$$
\begin{aligned}
G_1(\underline{u}) &= \sum_{i=0}^{\infty} \underline{A}_i(\underline{u}_0, \underline{u}_1, ..., \underline{u}_i), \quad G_1(\overline{u}) = \sum_{i=0}^{\infty} \overline{A}_i(\overline{u}_0, \overline{u}_1, ..., \overline{u}_i), \\
G_2(\underline{u}) &= \sum_{i=0}^{\infty} \underline{B}_i(\underline{u}_0, \underline{u}_1, ..., \underline{u}_i), \quad G_2(\overline{u}) = \sum_{i=0}^{\infty} \overline{B}_i(\overline{u}_0, \overline{u}_1, ..., \overline{u}_i), \\
D^j(\underline{u}) &= \sum_{i=0}^{\infty} \underline{L}_{i_j}(\underline{u}_0, \underline{u}_1, ..., \underline{u}_i), \quad D^j(\overline{u}) = \sum_{i=0}^{\infty} \overline{L}_{i_j}(\overline{u}_0, \overline{u}_1, ..., \overline{u}_i),
\end{aligned} \tag{6}
$$

where $A_i = (\underline{A}_i, \overline{A}_i)$, $B_i = (\underline{B}_i, \overline{B}_i)$, $L_{i_j} = (\underline{L}_{i_j}, \overline{L}_{i_j})$ at $i \geq 0$ are the so-called Adomian polynomials defined by

$$
\begin{aligned}
\underline{A}_i(\underline{u}_0, \underline{u}_1, ..., \underline{u}_i) &= \frac{1}{i!}\frac{d^i}{d\lambda^i}\left[G_1\left(\sum_{n=0}^{\infty} \lambda^n \underline{u}_n\right)\right]_{\lambda=0}, \\
\overline{A}_i(\overline{u}_0, \overline{u}_1, ..., \overline{u}_i) &= \frac{1}{i!}\frac{d^i}{d\lambda^i}\left[G_1\left(\sum_{n=0}^{\infty} \lambda^n \overline{u}_n\right)\right]_{\lambda=0}, \\
\underline{B}_i(\underline{u}_0, \underline{u}_1, ..., \underline{u}_i) &= \frac{1}{i!}\frac{d^i}{d\lambda^i}\left[G_2\left(\sum_{n=0}^{\infty} \lambda^n \underline{u}_n\right)\right]_{\lambda=0}, \\
\overline{B}_i(\overline{u}_0, \overline{u}_1, ..., \overline{u}_i) &= \frac{1}{i!}\frac{d^i}{d\lambda^i}\left[G_2\left(\sum_{n=0}^{\infty} \lambda^n \overline{u}_n\right)\right]_{\lambda=0}, \\
\underline{L}_{ij}(\underline{u}_0, \underline{u}_1, ..., \underline{u}_i) &= \frac{1}{i!}\frac{d^i}{d\lambda^i}\left[D^j\left(\sum_{n=0}^{\infty} \lambda^n \underline{u}_n\right)\right]_{\lambda=0}, \\
\overline{L}_{ij}(\overline{u}_0, \overline{u}_1, ..., \overline{u}_i) &= \frac{1}{i!}\frac{d^i}{d\lambda^i}\left[D^j\left(\sum_{n=0}^{\infty} \lambda^n \overline{u}_n\right)\right]_{\lambda=0}.
\end{aligned} \tag{7}
$$

The modified Adomian decomposition method is based on the assumption that the functions $\underline{G}(x,r)$ and $\overline{G}(x,r)$ may be divided into two parts, so

$$
\underline{G}(x,r) = \underline{G}_1(x,r) + \underline{G}_2(x,r),
$$
$$
\overline{G}(x,r) = \overline{G}_1(x,r) + \overline{G}_2(x,r),
$$

where

$$
\underline{G}_1(x,r) = \underline{b}_0(r), \quad \underline{G}_2(x,r) = \frac{1}{p_k}L^{-1}(\underline{g}(x,r)) + \sum_{j=1}^{k-1}\frac{1}{j!}(x-a)^j\underline{b}_j(r),
$$

$$
\overline{G}_1(x,r) = \overline{b}_0(r), \quad \overline{G}_2(x,r) = \frac{1}{p_k}L^{-1}(\underline{g}(x,r)) + \sum_{j=1}^{k-1}\frac{1}{j!}(x-a)^j\overline{b}_j(r).
$$

As a result, we get the recurrence formula

$$
\begin{aligned}
\underline{u}_0(x,r) &= \underline{b}_0(r),\\
\underline{u}_1(x,r) &= \tfrac{1}{p_k}L^{-1}(\underline{g}(x,r)) + \sum_{j=1}^{k-1}\tfrac{1}{j!}(x-a)^j\underline{b}_j(r)+\\
&+\tfrac{1}{p_k k!}\int_a^x (x-s)^k k_1(x,s)\underline{A}_0 ds + \tfrac{1}{p_k}\int_a^b L^{-1}(f_2(x))h_2(s)\underline{B}_0 ds-\\
&-\tfrac{1}{p_k(k-1)!}\sum_{j=0}^{k-1} p_j\int_a^x (x-s)^{(k-1)}\underline{L}_{0_j} ds\\
\ldots\\
\underline{u}_{i+1}(x,r) &= \tfrac{1}{p_k k!}\int_a^x (x-s)^k k_1(x,s)\underline{A}_i ds + \tfrac{1}{p_k}\int_a^b L^{-1}(f_2(x))h_2(s)\underline{B}_i ds-\\
&-\tfrac{1}{p_k(k-1)!}\sum_{j=0}^{k-1} p_j\int_a^x (x-s)^{(k-1)}\underline{L}_{i_j} ds
\end{aligned}
\tag{8}
$$

In **Subsection § 2.1.4** sufficiently good conditions for the existence and uniqueness of the solution of the equation have been found.

Let the following conditions be satisfied:
(i) $g \in C([a,b],E^1)$, $k_i \in C([a,b]\times[a,b],\mathbb{R}_+)$, $i=1,2$;
(ii) exists $L^i \geq 0$ and $L_j \geq 0$ such that $D(G_i(u),G_i(v)) \leq L^i D(u,v)$ and $D(D^j(u),D^j(v)) \leq L_j D(u,v)$ for all $u,v \in E^1$, $i=1,2$, $j=0,1,...,k-1$
(iii) $\alpha = (L^1M_1 + L^2M_2 + kLM)(b-a) < 1$, where

$$\left|\frac{k_1(x,s)(x-s)^k}{k!p_k}\right| \leq M_1,\quad \left|\frac{1}{p_k}L^{-1}(f_2(x))h_2(s)\right| \leq M_2,\quad \left|\frac{(x-s)^{(k-1)}p_j}{p_k(k-1)!}\right| \leq M_j,$$

$j=0,1,...,k-1$, $a \leq s \leq x \leq b$, $M = max|M_j|$, $L = max|L_j|$.

Theorem 2.1.1. *Let the conditions be met* $(i)-(iii)$*. Then the integral equations (3) and (4) have only one solution.*

In **Subsection § 2.1.5** he convergence of the method is proven and the error estimate between the exact and the approximate solution of the studied equation is obtained

Theorem 2.1.2. *The endless row* $\underline{u}(x,r) = \sum_{i=0}^{\infty}\underline{u}_i(x,r)$ *received from (3) is congruent if* $0<\alpha<1$ *and* $|\underline{u}_1(x,r)| < \infty$.

Theorem 2.1.3. *Let the conditions be met $(i)-(iii)$. Then the maximum absolute error of the solution (5) on the integral equations (3) and (4) is given by the inequalities*

$$\max_{x\in J}|\underline{u}(x,r)-\sum_{i=0}^{m}\underline{u}_i(x,r)|\leq\frac{\alpha^m b}{1-\alpha}(M_1\underline{\phi_1}+M_2\underline{\phi_2}+M\underline{\phi_3}),\tag{9}$$

$$\max_{x\in J}|\overline{u}(x,r)-\sum_{i=0}^{m}\overline{u}_i(x,r)|\leq\frac{\alpha^m b}{1-\alpha}(M_1\overline{\phi_1}+M_2\overline{\phi_2}+M\overline{\phi_3}),\tag{10}$$

In **Section § 2.2** the Sumudu fuzzy transform is defined. Some of its properties are given and it is applied to fuzzy derivatives.

In **Subsection § 2.2.1** a definition of fuzzy Sumudu transform and its inverse is given.

Definition 2.2.1. Let $w:\mathbb{R}_+\to E^1$ is a continuous fuzzy function and the function $e^{-x}\odot w(ux)$ is integrable in a nonproprietary sense in $\mathbb{R}_+$. Then

$$(FR)\int_0^{\infty}e^{-x}\odot w(ux)dx,$$

is called the Sumudu fuzzy transform and is denoted by

$$W(u)=S[w(x)]=(FR)\int_0^{\infty}e^{-x}\odot w(ux)dx,\tag{11}$$

for $u\in[-\sigma_1,\sigma_2]$, where the variable u is mapped to the variable x and $\sigma_1,\ \sigma_2>0$.

Definition 2.2.3. The fuzzy inverse Sumudu transform is given by the formula

$$S^{-1}\left[W(u)\right]=w(x)=\left(s^{-1}[\underline{W}(u,r)],s^{-1}[\overline{W}(u,r)]\right),\tag{12}$$

where

$$s^{-1}[\underline{W}(u,r)]=\frac{1}{2\pi\imath}\int_{\gamma-\imath\infty}^{\gamma+\imath\infty}e^{\frac{x}{u}}\underline{W}(u,r)du$$

$$s^{-1}[\overline{W}(u,r)] = \frac{1}{2\pi\imath} \int\limits_{\gamma-\imath\infty}^{\gamma+\imath\infty} e^{\frac{x}{u}} \overline{W}(u,r)du$$

For each $r \in [0,1]$ the functions $\underline{W}(u,r)$ and $\overline{W}(u,r)$ are analytic functions for each $Reu \geq \gamma$, where γ is a real constant that is chosen appropriately.

In **Subsection § 2.2.2** properties for Sumudu fuzzy transform are given.

Theorem 2.2.2. *Let $c_1,\ c_2$ are arbitrary constants. Then*

$$\begin{aligned} S[c_1 \odot f(x) \oplus c_2 \odot g(x)] \quad &= c_1 \odot S[f(x)] \oplus c_2 \odot S[g(x)] = \\ &= c_1 \odot F(u) \oplus c_2 \odot G(u). \end{aligned}$$

Theorem 2.2.3. *Let a and b are arbitrary constants. Then*

$$S[e^{ax} \odot f(x)] = \frac{1}{1-au} F\left(\frac{u}{1-au}\right).$$

In **Subsection § 2.2.3** fuzzy convolution is presented.

Definition 2.2.4. Let $k,\ w : \mathbb{R}_+ \to \mathbb{R}$ are fuzzy integrable functions. Then the fuzzy convolution of $k(x)$ and $w(x)$ is given by the equality

$$(k * w)(x) = (FR) \int\limits_0^x k(x-s) \odot w(s)ds. \tag{13}$$

The symbol $*$ means fuzzy convolution.

Theorem 2.2.6. *Let $k,\ w : \mathbb{R}_+ \to \mathbb{R}$ are fuzzy integrable functions for which the fuzzy Sumudu transform exists, i.e. $s[k(x)] = K(u)$ and $S[w(x)] = W(u)$. Then*

$$S[(k * w)(x)] = us[k(x)] \odot S[w(x)]. \tag{14}$$

In **Subsection § 2.2.4** basic properties of fuzzy Sumudu transform related to fuzzy derivatives are given.

Theorem 2.2.7. *Let $w : \mathbb{R} \to E^1$ is a continuous fuzzy function. The functions $e^{-x} \odot w(ux)$, $e^{-x} \odot w^{(n)}(ux)$ are integrable in a nonproprietary* *sense in* $\mathbb{R}_+$. *Then*

$$S\left[w^{(n)}(x)\right] = \frac{d^n}{dx^n}[S[w(x)]], \tag{15}$$

where $n \in \mathbb{N}$.

In **Section § 2.3** a fuzzy Sumudu decomposition method is constructed, which is a combination of the fuzzy Sumudu transform and the fuzzy Adomian decomposition method.

In **Subsection § 2.3.1** the FSDM task has been set up.

$$\begin{gathered} w^{(n)}(x) = g(x) \oplus (FR)\int_0^x k_1(x-s) \odot G_1(w(s))ds \oplus \\ \oplus (FR)\int_0^b k_2(x-s) \odot G_2(w(s))ds, \end{gathered} \tag{16}$$

with initial conditions

$$w^{(i)}(0) = b_i, \quad i = 0, 1, 2, ..., n-1, \tag{17}$$

where $k_1,\ k_2 : [0, b] \to \mathbb{R}$, $G_1,\ G_2 : E^1 \to E^1$ are continuous functions in E^1 and $g,\ w : [a, b] \to E^1$ are continuous fuzzy functions and $b_i \in E^1$, $i = 0, 1, ...n-1$, $b \in \mathbb{R}$.

In **Subsection § 2.3.2** he Sumudu fuzzy transform is applied for equation (16).

$$\begin{aligned} &\underline{w}_0(x,r) = \sum_{j=1}^{n} s^{-1}\left[v^{(n-j)}\underline{b}_{(n-j)}(r)\right] + \underline{g}(x,r), \\ &\underline{w}_{(i+1)}(x,r) = s^{-1}\left[v^{(n+1)}s[k_1(x)]s[\underline{A}_i]\right] + s^{-1}\left[v^{(n+1)}s[k_2(x)]s[\underline{B}_i]\right], \end{aligned} \tag{18}$$

$$\begin{aligned} &\overline{w}_0(x,r) = \sum_{j=1}^{n} s^{-1}\left[v^{(n-j)}\overline{b}_{(n-j)}(r)\right] + \overline{g}(x,r), \\ &\overline{w}_{(i+1)}(x,r) = s^{-1}\left[v^{(n+1)}s[k_1(x)]s[\overline{A}_i]\right] + s^{-1}\left[v^{(n+1)}s[k_2(x)]s[\overline{B}_i]\right]. \end{aligned} \tag{19}$$

Chapter 3. Natural's Fuzzy Transform for Solving LFVIDE

Chapter 3. consists of 3 paragraphs in which there are separate sections for greater clarity of the study.

In **Section § 3.1** the task has been set up for the linear fuzzy Volterra integro-differential equation (LFVIDE)

$$\int_0^x k_1(x-s)\odot w(s))ds \oplus \int_0^x k_2(x-s)\odot w^{(n)}(s))ds = g(x), \quad k_2(x-s)\neq 0 \tag{20}$$

with initial conditions

$$w^{(i)}(0) = b_i, \quad i = 0, 1, 2, ..., n-1, \tag{21}$$

where $k_1,\ k_2 : [a,b]\times[a,b] \to \mathbb{R}$, are continuous functions in E^1, $g,\ u : [a,b] \to E^1$ are continuous fuzzy functions and $b_i,\ (i = 0, 1, ..., n-1)$ are constants.

In **Section § 3.2** the Natural transform of the Fourier integral is derived and related to the Laplace and Sumudu transforms

In **Subsection § 3.2.1** a definition of the fuzzy Natural transformation and the relationship between them is given.

Definition 3.2.3. Let $w : \mathbb{R}_+ \to E^1$ is a continuous fuzzy function and the fuzzy function

$$e^{-sx}\odot w(ux)$$

is integrable in a nonproprietary sense in $\mathbb{R}_+$. Then

$$(FR)\int_0^\infty e^{-sx}\odot w(ux)dx$$

is called the fuzzy transform of Natural and is denoted by

$$W[s;u)] = N[w(x)] = (FR)\int_0^\infty e^{-sx}\odot w(ux)dx, \tag{22}$$

where s and u are variables of the transformation.

In **Subsection § 3.2.2** and given properties of the fuzzy transformation of Natural (FNT).

Theorem 3.2.4. *Let c_1, c_2 are arbitrary constants. Then*

$$\begin{aligned} N[c_1 \odot f(x) \oplus c_2 \odot g(x)] &= c_1 \odot N[f(x)] \oplus c_2 \odot N[g(x)] = \\ &= c_1 \odot F[s;u] \oplus c_2 \odot G[s;u)]. \end{aligned}$$

Theorem 3.2.5. *Let $f : \mathbb{R}_+ \to E^1$ is a fuzzy function for which $N[f(x)] = F[s;u]$. Then*

$$N[f(ax)] = \frac{1}{a} F[\frac{s}{a};u)],$$

where a is an arbitrary constant.

Theorem 3.2.6. *Let a and b is an arbitrary constant. Then*

$$N[e^{-ax} \odot f(x)] = F[s+a;u].$$

In **Subsection § 3.2.3** the fuzzy convolution is given.

Definition 3.2.6. Let $k, \quad w : \mathbb{R}_+ \to \mathbb{R}$ are fuzzy integrable functions. Then the fuzzy convolution of $k(x)$ and $w(x)$ is given by the equality

$$(k * w)(x) = (FR) \int_0^x k(x-\tau) \odot w(\tau) d\tau,$$

where the symbol $*$ stands for fuzzy convolution.

Theorem 3.2.9. *Let $k, \quad w : \mathbb{R}_+ \to \mathbb{R}$ are fuzzy functions for which the fuzzy transformation of Natural exists, i.e. $n[k(x)] = K[s;u]$ and $N[w(x)] = W[s;u]$. Then*

$$N[(k * w)(x,t)] = un[k(x)] \odot N[w(x)]. \tag{23}$$

In **Subsection § 3.2.4** new results related to PTNs for m-th order fuzzy derivatives are obtained.

Theorem 3.2.10. *Let $w : \mathbb{R}_+ \to E^1$ is a fuzzy function. For each $x > 0$ and $m \in \mathbb{N}$ exists continuously gH-derived from $(m-1)-$ order and exists*

$\frac{d^m w(x)}{dx^m}$. *Functions*

$$e^{-sx} \odot w(ux), \quad e^{-sx} \odot \frac{d^m w(ux)}{dx^m}$$

are integrable in a nonproprietary sense in $\mathbb{R}_+ \times \mathbb{R}_+$. *Then*

$$N\left[\frac{d^m w(x)}{dx^m}\right] = \frac{d^m}{dx^m} N[w(x)]. \tag{24}$$

In **Section § 3.3** Natural's fuzzy transform is applied to Volterra's fuzzy linear integro-differential equation.

$$\begin{aligned} N\left[(FR)\int_0^x k_1(x-\tau)\odot w(\tau)d\tau\right] \oplus N\left[(FR)\int_0^x k_2(x-\tau)\odot w^{(m)}(\tau)d\tau\right] = \\ = N[g(x)]. \end{aligned} \tag{25}$$

using fuzzy convolution (23) we receive

$$un[k_1(x)] \odot N[w(x)] \oplus un[k_2(x)] \odot N[w^{(m)}(x)] = N[g(x)]. \tag{26}$$

We apply Theorem 3.2.5 and from the initial conditions (21), we receive

$$un[k_1(x)]n[\underline{w}(x,r)] + un[k_2(x)]\left[\frac{s^m}{u^m} n[\overline{w}(x,r)] - \sum_{q=1}^{m} \frac{s^{(q-1)}}{u^q} \overline{b}_{(m-q)}(r)\right] = n[\underline{g}(x,r)]$$

$$un[k_1(x)]n[\overline{w}(x,r)] + un[k_2(x)]\left[\frac{s^m}{u^m} n[\overline{w}(x,r)] - \sum_{q=1}^{m} \frac{s^{(q-1)}}{u^q} \underline{b}_{(m-q)}(r)\right] = n[\underline{g}(x,r)].$$

Therefore

$$\begin{aligned} &u^m n[k_1(x)]n[\underline{w}(x,r)] + s^m n[k_2(x)]n[\overline{w}(x,r)] = \\ &= u^{(m-1)} n[\underline{g}(x,r)] + n[k_2(x)] \sum_{q=1}^{m} s^{(q-1)} u^{(m-q-1)} \overline{b}_{(m-q)}(r), \end{aligned} \tag{27}$$

$$\begin{aligned} &u^m n[k_1(x)]n[\overline{w}(x,r)] + s^m n[k_2(x)]n[\underline{w}(x,r)] = \\ &= u^{m-1} n[\overline{g}(x,r)] + n[k_2(x)] \sum_{q=1}^{m} s^{(q-1)} u^{(m-q-1)} \underline{b}_{(m-q)}(r), \end{aligned} \tag{28}$$

Chapter 4. Fuzzy transformations for solving LPF-VIDE

Chapter 4. consists of 3 paragraphs in which there are separate sections for greater clarity of the study.

In **Section § 4.1** the task has been set up for the linear partial fuzzy Volterra integro-differential equation (LPFVIDE).

$$\begin{gathered}\sum_{i=1}^{m} a_i \odot \frac{\partial^i w(x,t)}{\partial x^i} \oplus \sum_{j=1}^{n} b_j \odot \frac{\partial^j w(x,t)}{\partial t^j} \oplus c \odot w(x,t) = \\ = g(x,t) \oplus (FR) \int_0^t k(t-s) \odot w(x,s) ds,\end{gathered} \tag{29}$$

with initial conditions

$$\frac{\partial^i w(x,0)}{\partial t^j} = \psi_j(x), \quad i = 0,1,...,n-1, \tag{30}$$

in boundary conditions

$$\frac{\partial^i w(0,t)}{\partial x^i} = \varphi_i(t), \quad i = 0,1,...,m-1, \tag{31}$$

where $k : [0,d] \to \mathbf{R}$ is a continuous function, $g,\ w : [0,b] \times [0,d] \to E^1$, $\varphi_i : [0,d] \to E^1$, $\psi_j : [0,b] \to E^1$ are continuous fuzzy functions and a_i, $i = 1,2,...m$, b_j, $j = 1,2,...n$, c, are constants.

In **Section § 4.2** the fuzzy Sumudu transform (FST) was used to solve fuzzy partial integro-differential equations.

In **Subsection § 4.2.1** a definition of RTS for a function of two variables and its inverse are given. In addition, some of its main properties have been proven.

Definition 4.2.1. Let $w : \mathbb{R}_+ \times \mathbb{R}_+ \to E^1$ is a continuous fuzzy function and the function $e^{-t} \odot w(x,vt)$ is integrable with respect to t in $\mathbb{R}_+$. Then

$$(FR) \int_0^\infty e^{-t} \odot w(x,vt) dt,$$

is called the fuzzy Sumudu transform for a function of two variables and is denoted by

$$W(x,v) = S_t[w(x,t)] = (FR)\int\limits_0^\infty e^{-t} \odot w(x,vt)dt, \tag{32}$$

for $v \in [-\sigma_1, \sigma_2]$, where the variable v is mapped to the variable t in the fuzzy function and $\sigma_1,\ \sigma_2 > 0$.

Definition 4.2.3. The fuzzy inverse Sumudu transform for a function of two variables is given by the formula

$$S_t^{-1}\left[W(x,v)\right] = w(x,t) = \left(s_t^{-1}[\underline{W}(x,v,r)], s_t^{-1}[\overline{W}(x,v,r)]\right), \tag{33}$$

where

$$s_t^{-1}[\underline{W}(x,v,r)] = \frac{1}{2\pi\imath}\int\limits_{\delta-\imath\infty}^{\delta+\imath\infty} e^{\frac{t}{v}}\,\underline{W}(x,v,r)dv$$

$$s_t^{-1}[\overline{W}(x,v,r)] = \frac{1}{2\pi\imath}\int\limits_{\delta-\imath\infty}^{\delta+\imath\infty} e^{\frac{t}{v}}\,\overline{W}(x,v,r)dv$$

For each $r \in [0,1]$ functions $\underline{W}(x,v,r)$ and $\overline{W}(x,v,r)$ are analytic functions for each $Rev \geq \delta$, where δ is a real constant that is chosen appropriately.

Theorem 4.2.2. *Let $c_1,\ c_2$ are arbitrary constants. Then*

$$\begin{aligned} S_t[c_1 \odot f(x,t) \oplus c_2 \odot g(x,t)] &= c_1 \odot S_t[f(x,t)] \oplus c_2 \odot S_t[g(x,t)] = \\ &= c_1 \odot F(x,v) \oplus c_2 \odot G(x,v). \end{aligned}$$

Theorem 4.2.3. *Let a is an arbitrary constant. Then*

$$S_t[e^{bt} \odot f(x,t)] = \frac{1}{1-bv} F\left(x, \frac{v}{1-bv}\right).$$

In**Subsection § 4.2.2** definition and theorem of fuzzy convolution are given.

Definition 4.2.4. Let $k(t)$ and $w(x,t)$ are fuzzy integrable functions. Then the fuzzy convolution of $k(t)$ and $w(x,t)$ regarding t is given by the equality

$$(k * w)(x,t) = (FR)\int_0^t k(t-s) \odot w(x,s)ds. \tag{34}$$

where the symbol $*$ means the fuzzy convolution about t.

Theorem 4.2.6. *Let $k : \mathbb{R}_+ \to \mathbb{R}$ and $w : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ are fuzzy functions for which the fuzzy two-dimensional Sumudu transform exists, i.e. $s_t[k(t)] = K(v)$ and $S_t[w(x,t)] = W(x,v)$. Then*

$$S_t[(k * w)(x,t)] = v s_t[k(t)] \odot S_t[w(x,t)]. \tag{35}$$

In **Subsection § 4.2.3** the main properties of the FST for a function of two variables related to partial fuzzy derivatives are obtained.

Theorem 4.2.7. *Let $w : \mathbb{R}_+ \times \mathbb{R}_+ \to E^1$ is a continuous function. The functions*

$$e^{-t} \odot w(x,vt), \quad e^{-t} \odot \frac{\partial^n w(x,vt)}{\partial t^n}$$

are integrable about t in $\mathbb{R}_+$. Then

$$S_t\left[\frac{\partial^n w(x,t)}{\partial t^n}\right] = \frac{\partial^n}{\partial t^n} S_t[w(x,t)], \tag{36}$$

where $n \in \mathbb{N}$.

In **Subsection § 4.2.3** the FST method is used for the investigated equation, which reduces to a fuzzy ordinary differential equation.

We apply FST about the variable t and we get

$$S\left[\sum_{i=1}^{m} a_i \frac{\partial^i w(x,t)}{\partial x^i}\right] \oplus S\left[\sum_{j=1}^{n} b_j \frac{\partial^j w(x,t)}{\partial t^j}\right] \oplus S[c \odot w(x,t)] =$$
$$= S[g(x,t)] \oplus S\left[(FR)\int_0^t k(t-s) \odot w(x,s)ds\right],$$

We use the initial conditions (30) and we obtain a system of ordinary differential equations from m-th row.

$$\sum_{i=1}^{m} a_i \frac{d^i \underline{W}(x,v,r)}{dx^i} + \left(\sum_{j=1}^{n} \frac{b_j}{v^j} + c - vs[k(t)] \right) \underline{W}(x,v,r)] =$$
$$= s[\underline{g}(x,t,r)] + \sum_{j=1}^{n} \sum_{k=1}^{j} \frac{b_j}{v^k} \underline{\psi}_{j-k}(x,r),$$

$$\sum_{i=1}^{m} a_i \frac{d^i \overline{W}(x,v,r)}{dx^i} + \left(\sum_{j=1}^{n} \frac{b_j}{v^j} + c - vs[k(t)] \right) \overline{W}(x,v,r)] =$$
$$= s[\overline{g}(x,y,r)] + \sum_{j=1}^{n} \sum_{k=1}^{j} \frac{b_j}{v^k} \underline{\psi}_{j-k}(x,r).$$

In **Section § 4.3** the Natural fuzzy transform (NFT) for solving fuzzy partial integro-differential equations is studied.

In **Subsection § 4.3.1** definition of fuzzy two-dimensional Natural transform (TDNT) of a function of two variables and its inverse are given.

Definition 4.3.1. [15] Let $w : \mathbb{R}_+ \times \mathbb{R}_+ \to E^1$ is a continuous fuzzy function. We assume that the fuzzy function

$$e^{-(sx+pt)} \odot w(ux, vt)$$

is integrable in a nonproprietary sense in $\mathbb{R}_+ \times \mathbb{R}_+$. Then

$$(FR)\int_0^\infty (FR)\int_0^\infty e^{-(sx+pt)} \odot w(ux, vt)dxdt$$

is called the fuzzy two-dimensional Natural transform and is denoted by

$$W[(s,p);(u,v)] = N[w(x,t)] = (FR)\int_0^\infty (FR)\int_0^\infty e^{-(sx+pt)} \odot w(ux, vt)dxdt, \tag{37}$$

where $s,\ p > 0$ and $u,\ v > 0$ are variables of the transformation.

Definition 4.3.2. [15] The fuzzy two-dimensional inverse Natural transform is given by the formula

$$N^{-1}\left[W[(s,p);(u,v)]\right] = w(x,y) =$$
$$= \left(n^{-1}[\underline{W}[(s,p);(u,v,r)]], n^{-1}[\overline{W}[(s,p);(u,v,r)]]\right),$$

where

$$n^{-1}[\underline{W}[(s,p);(u,v,r)]] = \frac{1}{2\pi\imath}\int\limits_{\gamma-\imath\infty}^{\gamma+\imath\infty} e^{\frac{sx}{u}} du \frac{1}{2\pi\imath}\int\limits_{\delta-\imath\infty}^{\delta+\imath\infty} e^{\frac{py}{v}} \underline{W}[(s,p);(u,v,r)]dv,$$

and

$$n^{-1}[\overline{W}[(s,p);(u,v,r)]] = \frac{1}{2\pi\imath}\int\limits_{\gamma-\imath\infty}^{\gamma+\imath\infty} e^{\frac{sx}{u}} du \frac{1}{2\pi\imath}\int\limits_{\delta-\imath\infty}^{\delta+\imath\infty} e^{\frac{py}{v}} \overline{W}[(s,p);(u,v,r)]dv.$$

For each $r \in [0,1]$ functions $\underline{W}[(s,p);(u,v,r)]$ and $\overline{W}[(s,p);(u,v,r)]$ are analytic functions for each $Reu \geq \gamma$ and $Rev \geq \delta$, where γ and δ are real constants that are appropriately chosen.

In **Subsection § 4.3.2** basic properties of DFNT are given.

Theorem 4.3.2. *Let c_1, c_2 are arbitrary constants. Then*

$$\begin{aligned} N[c_1 \odot f(x,t) \oplus c_2 \odot g(x,t)] &= c_1 \odot N[f(x,t)] \oplus c_2 \odot N[g(x,t)] = \\ &= c_1 \odot F[(s,p),(u,v)] \oplus c_2 \odot G[(s,p),(u,v)]. \end{aligned}$$

Theorem 4.3.4. *Let a and b are arbitrary constants. Then*

$$N[e^{(-ax-bt)} \odot f(x,t)] = F[(s+a,p+b);(u,v)].$$

In **Subsection § 4.3.3** definition and theorem of fuzzy convolution are given.

Definition 4.3.4. Let $k(t)$ and $w(x,t)$ are fuzzy integrable functions. Then the fuzzy convolution of $k(t)$ and $w(x,t)$ regarding t is given by the equality

$$(k * w)(x,t) = (FR)\int\limits_0^t k(t-s) \odot w(x,s)ds,$$

where the symbol $*$ means the fuzzy convolution about t.

Theorem 4.3.7. *Let $k : \mathbb{R}_+ \to \mathbb{R}$ and $w : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ are fuzzy functions for which the double fuzzy Natural transformation exists, i.e. $n[k(t)] = K[p;v]$ and $N[w(x,t)] = W[(s,p);(u,v)]$. Then*

$$N[(k * w)(x,t)] = vn[k(t)] \odot N[w(x,t)]. \tag{38}$$

In **Subsection § 4.3.4** new results for DFNT fuzzy partial derivatives of the m-th order are obtained.

Theorem 4.3.8. *Let $w : \mathbb{R}_+ \times \mathbb{R}_+ \to E^1$ is a fuzzy function. For each $x > 0$ and $m \in \mathbb{N}$ there exists a continuous private gH-derivative of $(m-1)-th$ row regarding x and exists $\frac{\partial^m w(x,t)}{\partial x^m}$. The functions*

$$e^{-(sx+pt)} \odot w(ux, vt), \quad e^{-(sx+pt)} \odot \frac{\partial^m w(ux, vt)}{\partial x^m}$$

are integrable in a nonproprietary sense in $\mathbb{R}_+ \times \mathbb{R}_+$. Then

$$N\left[\frac{\partial^m w(x,t)}{\partial x^m}\right] = \frac{\partial^m}{\partial x^m} N[w(x,t)]. \tag{39}$$

In **Subsection § 4.3.5** the double Natural fuzzy transform is used to find the exact solution of a LVPIDE
We apply DFNT for equation (29) and obtain

$$N\left[\sum_{i=1}^{m} a_i \frac{\partial^i w(x,y)}{\partial x^i}\right] \oplus N\left[\sum_{j=1}^{k} b_j \frac{\partial^j w(x,y)}{\partial y^j}\right] \oplus N[c \odot w(x,y)] =$$

$$= N[g(x,y)] \oplus N\left[(FR)\int_0^t k(t-s) \odot w(x,s)ds\right].$$

We use fuzzy convolution () and we find

$$N\left[\sum_{i=1}^{m} a_i \frac{\partial^i w(x,y)}{\partial x^i}\right] \oplus N\left[\sum_{j=1}^{k} b_j \frac{\partial^j w(x,y)}{\partial y^j}\right] \oplus N[c \odot w(x,y)] =$$
$$= N[g(x,y)] \oplus vN[k(t)] \odot N[w(x,t)].$$

From the fuzzy partial derivatives theorems of the m-th order and the initial conditions, we obtain

$$\left(\sum_{i=1}^{m} \frac{a_i s^i}{u^i} + \sum_{j=1}^{k} \frac{b_j p^j}{v^j} + vn[k(t)] + c\right) n[\underline{w}(x,t,r)] =$$
$$= n[\underline{g}(x,t,r)] + \sum_{i=1}^{m}\sum_{q=1}^{i} \frac{a_i s^{(q-1)}}{u^q} n\left[\underline{\varphi}^{(i-q)}(t,r)\right] + \sum_{j=1}^{k}\sum_{q=1}^{j} \frac{b_j p^{(q-1)}}{v^q} n\left[\underline{\psi}^{(j-q)}(x,r)\right],$$

$$\left(\sum_{i=1}^{m} \frac{a_i s^i}{u^i} + \sum_{j=1}^{k} \frac{b_j p^j}{v^j} + vn[k(t)] + c\right) n[\overline{w}(x,t,r)] =$$
$$= n[\overline{g}(x,t,r)] + \sum_{i=1}^{m} \sum_{q=1}^{i} \frac{a_i s^{(q-1)}}{u^q} n\left[\overline{\varphi}^{(i-q)}(t,r)\right] + \sum_{j=1}^{k} \sum_{q=1}^{j} \frac{b_j p^{(q-1)}}{v^q} n\left[\overline{\psi}^{(j-q)}(x,r)\right].$$

Conclusion

Summary of the results obtained

In the opinion of the author, the main contributions in this dissertation work are:

1. Sufficient conditions for the existence and uniqueness of the solution of a nonlinear fuzzy Voltera-Fredholm integro-differential equation are found.
2. A fuzzy analytical method using the Adomian decomposition method is constructed to find the approximate solution of a nonlinear fuzzy Volterra-Fredholm integro-differential equation. Sufficient conditions for the convergence of the method are found and an estimate of the error is obtained.
3. A fuzzy transform of Sumudu is constructed. Sufficient conditions for the existence of the transformation and its application to ordinary and partial fuzzy derivatives are found.
4. A fuzzy analytical method is constructed which is a combination of Sumudu fuzzy transform and Adomian decomposition method to find the approximate solution for the nonlinear Volterra-Fredholm fuzzy integro-differential equation.
5. A fuzzy Natural transform is constructed to find the exact solution of a linear fuzzy Volterra integro-differential equation with a convolutional kernel. Sufficient conditions for the existence of the transformation and its connection with the Laplace and Sumudu transformations are found.
6. A fuzzy analytical method is constructed that uses the fuzzy variant of the Sumudu transform to find the exact solution of a linear partial fuzzy Volterra integro-differential equation.
7. A fuzzy two-dimensional Natural transform is constructed to find the exact solution of a linear partial fuzzy Volterra integro-differential equation. Sufficient conditions for the existence of the transformation and its application for fuzzy partial derivatives are found.

The relationships between contributions, goals, tasks, the place of description in the dissertation, publications and reports on the topic are as follows:

Contributions	Purpose	Tasks	Sections	Publications	Reports
1	1	a	2.1.4	1	1
2	2	b	2.1.3, 2.1.4, 2.1.5	3	3
3	3	c	2.2, 4.2	2, 3	2, 3
4	2	d	2.3	3	3
5	3	e	3.2, 3.3		5
6	3	f	4.2	2	2
7	3	g	4.3	4	4

List of publications

The main results of this dissertation have been published and cited in the following scientific articles:

1. **Georgieva A., Spasova M.**, Solving nonlinear Volterra-Fredholm fuzzy integro-differential equations by using Adomian decomposition method, *AIP Conference Proceedings 2333, 080005 (2021)*,
View online: https://doi.org/10.1063/5.0041602,
(Web of Science, SJR=0.189, 2021).
2. **Georgieva .A, Spasova M.**, Solving partial fuzzy integro-differential equations using fuzzy Sumudu transform method, *AIP Conference Proceedings 2321, 030010 (2021)*; View online: https://doi.org/10.1063/5.0040137, (Web of Science, SJR=0.189, 2021).
3. **Georgieva A., Spasova M.**, Sumudu decomposition method for solving Volterra-Fredholm fuzzy integro-differential equations, *AIP Conference Proceedings 2505, 070003 (2022)*,
View online: https://doi.org/10.1063/5.0100648.,
(Web of Science, SJR=0.164, 2022).
4. **Georgieva A., Spasova M.**, Solution of partial fuzzy integro-differential equations by double natural transform, *AIP Conference Proceedings 2459, 030012 (2022)*, View online: https://doi.org/10.1063/5.0083628, (Web of Science, SJR=0.164, 2022).

Quotes:

a. **Jamal N.,Sarwar M., Agarwal P., Mlaiki N., Aloqaily A.**, Solutions of fuzzy advection-difusion and heat equations by natural Adomian

decomposition method, *International Journal of Legal Medicine* Sci Rep 13, 18565, (2023), View online: https://https://doi.org/10.1038/s41598-023-45207-y (IF = 2.1, 2022).

Approbation of the obtained results

A) Reports of international conferences

1. **Georgieva A., Spasova M.**,
Solving nonlinear Volterra-Fredholm fuzzy integro-differential equations by using Adomian decomposition method, 46 International Conference Applications of Mathematics in Ingenering and Economics (AMEE'20), Sozopol, Bulgaria, 8 - 13 June 2020.
2. **Georgieva A., Spasova M.**,
Solving partial fuzzy integro-differential equations using fuzzy Sumudu transform method, 7th International Conference New Trends in the Applications of Differential Equations in Sciences (NTADES'20), St. Konstantin and Elena, Bulgaria, 1-4 September 2020.
3. **Georgieva A., Spasova M.**,
Sumudu decomposition method for solving Volterra-Fredholm fuzzy integro-differential equations, 47 International Conference Applications of Mathematics in Ingenering and Economics (AMEE'21), Sozopol, Bulgaria, 7 - 13 June 2021.
4. **Georgieva A., Spasova M.**,
Solution of partial fuzzy integro-differential equations by double natural transform, 8th International Conference New Trends in the Applications of Differential Equations in Sciences (NTADES'21), St. Konstantin and Elena, Bulgaria, 6 - 9 September 2021.
5. **Georgieva A., Spasova M.**,
Fuzzy Sumudu Transform to Solve Convolution Type Volterra Fuzzy Integro-Differential Equations, Second International E-Conference on Mathematical and Statistical Science: A Selcuk Meeting (ICOMSS'23), Selcuk, Turkiye, 5 - 7 June 2023.

B) Participation in projects

1. Research project Modern studies of some classes of differential and differential equations: KP-06-N32/7.

2. Science project MU19-FMI-009 to Division ”Scientific and Project Activity” of PU on topic: “Modeling through mathematics and informatics and their symbiosis with ICT”, 2019-2020
3. Science project MU21-FMI-007 to Division ”Scientific and Project Activity” of PU on topic: “Symbiosis between mathematics and informatics (SMI in FMI)”, 2021-2022

Declaration of originality

From **Mira Lachezarova Spasova**,
regular PhD student at the Department of "Mathematical Analysis"
at the Faculty of Mathematics and Informatics
of Plovdiv University "Paisiy Hilendarski"

n connection with the procedure for the acquisition of the educational and scientific degree "doctor" at the Paisii Hilendarski University of Plovdiv and the defense of the dissertation work presented by me, I declare:

The results and contributions of the conducted dissertation research, presented in my dissertation on "Analytical methods for solving some classes of fuzzy integro-differential equations" are original and not borrowed from research and publications in which I have no involvement.

10.02.2024 DECLARATOR:
Plovdiv /Mira Spasova/

Bibliography

[1] **Abbasbandy S., Hashemi M.**, *A series solution of fuzzy integro-differential equations*, J. Fuzzy Set Valued Anal., 00066, (2012).

[2] **Ahmad J., Nosher H.**, *Solution of Different Types of Fuzzy Integro-Differential Equations Via Laplace Homotopy Perturbation Method*, J. Sci. Arts, 17, 5, (2017).

[3] **Ali M.R.**, *Approximate solutions for fuzzy Volterra integro-differential equations*, Journal of Abstract and Computational Mathematics, 3(1):11–22, (2018).

[4] **Anastassiou G. A.**, *Fuzzy Mathematics ,Approximation Teory*, Springer, Berlin, Germany, (2010).

[5] **Behzadi S. S., Allahviranloo T., Abbasbandy S.**, *Solving fuzzy second-order nonlinear Volterra-Fredholm integro-differential equations by using Picard method*, Neural Computing and Applications, 21, 337-346, (2012).

[6] **Biswas, S., Roy, T. K.**, *Generalization of Seikkala derivative and differential transform method for fuzzy Volterra integro - differential equations*, Journal of Intelligent and Fuzzy Systems, 34(4):2795–2806, (2018).

[7] **Bo T. L., Xie L., Zheng X. J.**, *Numerical approach to wind ripple in desert*, Int. J. Nonlinear Sci. Numer. Simul., 8:223–228, (2007).

[8] **Buckley J., Eslami E., Feuring T.**, *Fuzzy Integral Equations, in: Fuzzy Mathematics in Economics and Engineering*, Studies in Fuzziness and Soft Computing, Springer,Physica-Verlag, Heidelberg, vol.91, 229-241, (2002).

[9] **Bushnaq S., Ullah Z., Ullah A., Shah, K.**, *Solution of fuzzy singular integral equation with Abel's type kernel using a novel hybrid method*, Advances in Difference Equations, (1), 13, (2020).

[10] **Chalco-Cano Y., Roman-Flores H.**, *On New Solutions of Fuzzy Differential Equations*, Chaos, Solitons and Fractals, 38:112-119, (2008).

[11] **Chalishajar D., Ramesh R.**, *Controllability for impulsive fuzzy neutral functional integro-differential equations*, AIP Conf. Proc., 12159, 030007, (2019).

[12] **Georgieva A.**, *Double Fuzzy Sumudu Transform to Solve Partial Volterra Fuzzy Integro-Differential Equations, Mathematics*, 8:692, (2020).

[13] **Georgieva A.**, *Application of double fuzzy Natural transform for solving fuzzy partial equations*, AIP Conf. Proc., 2333, 080006-1–080006-8, (2021).

[14] **Georgieva A., Alidema A.**, *Adomian decomposition method for solving two-dimensional nonlinear Volterra fuzzy integral equations*, AIP Conference Proceedings. 2048, 050009, (2018).

[15] **Georgieva A., Spasova M.**, *Solving partial fuzzy integro-differential equations using fuzzy Sumudu transform method*, AIP Conference Proceedings. 2321, 030010, (2021).

[16] **Georgieva A., Spasova M.**, *Solving nonlinear Volterra-Fredholm fuzzy integro-differential equations by using Adomian decomposition method*, AIP Conference Proceedings, 2333, 080005, (2021).

[17] **Georgieva A., Spasova M.**, *Solution of partial fuzzy integro-differential equations by double natural transform*, AIP Conference Proceedings 2459, 030012, (2022).

[18] **Georgieva A., Spasova M.**, *Sumudu decomposition method for solving Volterra-Fredholm fuzzy integro-differential equations*, AIP Conference Proceedings 2505, 070003, (2022).

[19] **Hooshangian L.**, *Nonlinear Fuzzy Volterra Integro-differential Equation of N-th Order: Analytic Solution and Existence and Uniqueness of Solution*, Int. J. Industrial Mathematics, 11, 1, (2019).

[20] **Xu L., He J. H., Liu Y.**, *Electrospun nanoporous spheres with Chinese drug*, nt. J. Nonlinear Sci. Numer. Simul., 8:199–202, (2007).

[21] **Kang, S. M., Iqbal, Z., Habib, M., Nazeer, W.**, *Sumudu decomposition method for solving fuzzy integro-differential equations*, Axioms, 8(2):74–18, (2019).

[22] **Majid Z., Rabiei F., Hamid F., Ismail F.**, *Fuzzy Volterra Integro-Differential Equations Using General Linear Method*, Symmetry, 11:381-291, (2019).

[23] **Mikaeil, N., Khakrangin S., Allahviranloo T.**, *Solving fuzzy Volterra integro-differential equation by fuzzy differential transform method*, In Proceedings of the 7th Conference of the European Society for Fuzzy Logic and Technology, Aix-Les-Bains, France, 18–22 July (2011).

[24] **Sekar S., Thirumurugan A. S.**, *Numerical investigation of the fuzzy integro - differential equations by He's Homotopy perturbation method*, Malaya Journal of Matematik, 7(3):468–471, (2019).

[25] **Shakeri F., Dehghan M.**, *Solution of a model describing biological species living together using the variational iteration method*, Math. Comput. Model., 48:685–699, (2008).

[26] **Sindu Devi, S., Ganesan, K.**, *Fuzzy Picard's method for derivatives of second order*, Journal of Physics, 1–7, (2018).

[27] **Smithson M. J., Verkuilen J.,**, *Applications in the Social Sciences. Quantitative Applications in the Social Sciences*, Fuzzy Set Theory: 112, (2006).

[28] **Sun F. Z., Gao M., Lei S. H., Zhao Y. B., Wang K., Shi Y. T., Wang, N. H.**, *The fractal dimension of the fractal model of dropwise condensation and its experimental study*, Int. J. Nonlinear Sci. Numer. Simul., 8:211–222,(2007).

[29] **Syropoulos A., Grammenos T.**, *A Modern Introduction to Fuzzy Mathematics, John Wiley and Sons*, 384 Pages, ISBN: 978-1-119-44528-9, (2020).

[30] **Tunc C., Tunc O.**, *New qualitative criteria for solutions of Volterra integro-differential equations*, Arab Journal of Basic and Applied Sciences, 25(3):158–165, (2018).

[31] **Wang H., Fu H. M., Zhang H. F., Hu Z. Q.**, *A practical thermodynamic method to calculate the best glass-forming compositio for bulk metallic glasses*, Int. J. Nonlinear Sci. Numer. Simul., 8:171–178, (2007).

[32] **Wasques V.**, *Fuzzy Differential Equations via Interactive Arithmetic: Applications in Biomathematics*, Universidade Estadual de Campinas, Instituto de Matematica, Estatistica e Computacao Cientifica, 219, (2019).

[33] **Watugala K. G.**, *Sumudu transform a new integral transform to solve differential equations and control engineering problems*, Internat. J. Math. Ed. Sci. Tech., 24:35-43, (1993).

[34] **Bellman R., Zadeh L.**, *Questions analysis and procedures of acceptance and decisions*, M., MIR, (1976).

[35] **Orlovsky S.**, *Decision-making problems with fuzzy initial information*, Science Moscow, pp 207., (1984).

Acknowledgments

I express my heartfelt gratitude to my supervisor Prof. Dr. Atanaska Georgieva for the acquired knowledge, skills, support and patience she gave me in the development and shaping of the dissertation work.

Acknowledgments

The results in the dissertation work are part of the research on several projects, one of which is:

Project № KP-06 PN 62/1,Mathematical and informational modeling of dynamic processes - new theoretical results, research methods and applications, Prof. Dr. Snezhana Hristova, 2022 - (active).