R E V I E W

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on a competition for occupying the academic position of **Associate Professor**

area of higher education: *4. Natural Sciences, Mathematics and Informatics*,
professional field *4.5 Mathematics*,
scientific speciality *Geometry and Topology*

In the competition for occupying the academic position "Associate Professor" announced in the State Gazette No. 96 of November 11, 2023 and on the website of the Plovdiv University "Paisii Hilendarski" for the needs of the Department "Algebra and Geometry" at the Faculty of Mathematics and Informatics, documents have been submitted only by Assistant Professor Dr. Iva Dokuzova, member of the Department "Algebra and Geometry", FMI, PU.

1. General presentation of the received materials

Subject

I am a member of the scientific panel for this procedure according to order No. RD-21-387/16.02.2024 of the Rector of the Plovdiv University "Paisii Hilendarski" Only one candidate has submitted documents for participation in the announced competition, Assistant Professor Dr. Iva Rumenova Dokuzova, Department of Algebra and Geometry, FMI, PU.

I have received the set of documents submitted by Assistant Professor Dr. Iva Dokuzoza for participation in the competition via e-mail. These documents are in accordance with the Act on the Development of the Academic Staff in the Republic of Bulgaria and the relevant Regulations of Plovdiv University.

Dr. Iva Rumenova Dokuzova has submitted 15 scientific works, one monograph, a reference for citations of her works, two textbooks, a reference for participation in 3 scientific and educational projects, a reference for her educational activities and scientific supervision of two successfully defended graduates, as well as other documents required by the Law and Regulations.

I will consider 15 scientific works by Dr.Dokuzova which have not been used in her Ph.D. thesis and for occupying the academic position of Chief Assistant Professor, as well as the presented monograph. Furthermore, the 2 textbooks, the candidate's participation in research and educational projects and her work with students will be taken into account in the final assessment.

The candidate has presented a list of a total of 22 scientific articles. It what follows I accept for consideration the 15 of them submitted for the competition.

The distribution of the scientific works submitted for the participation in the competition, according to the rubrics in the country and abroad, is as follows: 7 works in the country, 8 - abroad. A list of a total of 41 citations of the candidate's works is presented, 29 of them of publications for participation in the competition.

2. Personal data

Dr. Iva Rumenova Dokuzova was born in 1971. She completed her higher education with master's degree at the Faculty of Mathematics and Informatics of Plovdiv University in 1994. After that, for one year, 1995 - 1996, she was a high school teacher in Samokov. In the period 1998 - 2009, she was successively assistant, senior assistant, chief assistant at PU ”Paisii Hilendarski”, Branch ”Lyuben Karavelov”, Kardzhali. In 2006, she successfully defended a dissertation for obtaining the scientific and educational degree ”Doctor”, scientific specialty ”Geometry and Topology” with a thesis in the field of differential geometry. From 2009 to the present, she is a Chief Assistant Professor at the Plovidv University ”Paisii Hilendarski”, Faculty of Mathematics and Informatics, Department of "Algebra and Geometry".

3. General characteristics of the applicant's activity

Educational and pedagogical activity

My assessment of the candidate's educational and pedagogical activity is positive. It is based on the presented reference for her work with students, the reference for the candidate's classroom and non-auditory employment, the lecture courses and study aids developed by her, as well as her two textbooks presented (one of them with a co-author).

Participation in research projects

Dr. Iva Rumenova Dokuzova has participated in 3 research projects - two funded by the PU and one funded under the operational program ”Science and Education for Intelligent Growth” of EU.

Scientometric data

As I mentioned, Dr. Iva Dokuzova participated in the competition with 15 papers, all in English. Of them, 11 are in scientific journals, 4 - in proceedings of national conferences; 3 papers are in journals with Impact Factor, 5 are in journals with SJR, 3 papers in journals without IF or SJR, but referenced in Mathematical Reviews or Zentralblatt für Mathematik, and 4 papers do not fall into these categories.

Nine of the papers are personal work of the candidate, two have one co-author, four have two co-authors. I accept that the candidate's contribution to the joint publications is equal to the other co-authors. The monograph is a personal work of the candidate.

I would like also to mention that I have not discovered any plagiarism.

The candidate has submitted a reference for 41 citations of her works, 29 of which refer to papers submitted for participation in the competition; these are the citing articles numbered 9 - 35 and 37, 38 in the list of citations. Of the total number of citations, 5 citing papers are in journals with IF, 5 - with SJR, 7 - in journals indexed in Scopus, 2- are referenced in MR or Zbl.

4. Review of result obtained by the candidate

Papers numbered 1 and 3 in the list of competition publications deal with an almost product structure J on a smooth manifold M compatible with a Riemannian or pseudo- Riemannian metric g, in the sense that $g(JX, JY) = g(X, Y)$. This compatibility condition is analogous to the condition that an almost complex structure is compatible with a Riemannian metric, i.e., the manifold is almost Hermitian, but the difference is essential. In the case of a compatible almost product structure, the fundamental bilinear form $\omega(X, Y) = g(JX, Y)$ is symmetric, while for a compatible almost complex structure it is skew-symmetric. Given vector fields a and b on M, using the Levi-Civita connection and the vector fields a and b, a new symmetric connection $\overline{\nabla}$ (which, of course, is not metric) is defined. In a private conversation, the candidate has explained that the idea of introducing the new connection as a transformation of Levi-Civita connection, broadly speaking, had came from papers on holomorphic projective transformations and from results in her doctoral dissertation on almost complex manifolds with a compatibility condition $g(JX, JY) = -g(X, Y)$. It is proved that the new connection satisfies the identity

$$(\overline{\nabla}_X \omega)(Y, Z) + (\overline{\nabla}_Y \omega)(Z, X) + (\overline{\nabla}_Z \omega)(X, Y) = 0, \tag{1}$$

provided the Levi-Civita connection ∇ satisfies an analogous identity. Going back to the analogy with the almost Hermitian case, one can say that identity (1) for the Levi-Civita connection is analogous to the condition that the almost Hermitian manifold is almost Kähler, i.e., its second fundamental form ω to be closed: $d\omega = 0$. In papers 1 and 3, relations between geometric properties of the connections $\overline{\nabla}$ and ∇ are established and geometric characteristics of $\overline{\nabla}$ (curvature tensor, Ricci tensor) are calculated in the case when the vector field a is zero.

The geometric objects and problems considered in paper 2 are similar to those in papers 1 and 3, but in the case of a manifold endowed with an almost complex structure J and a pseudo-Riemannian metric g with compatibility condition $g(JX, JY) = -g(X, Y)$. In this case, the manifolds for which the Levi-Civita connection and the form $\omega(X, Y) = g(JX, Y)$ satisfy identity (1) are called quasi-Kähler by Borisov and Ganchev (1986). For comparison, the quasi-Kähler almost Hermitian manifolds are defined by the identity $(\nabla_X \omega)(Y, Z) + (\nabla_{JX} \omega)(JY, Z) = 0$ (Gray, Hervella, 1980). However, the form ω is skew-symmetric for almost Hermitian manifolds, while in the pseudo-Riemannian case it is symmetric. Note also that the metric g and the non-degenerate symmetric form ω are of neutral signature (n, n) (the form ω is frequently called an associated metric).

Although in paper 4, as well as in other of the presented papers, it is stated that three-dimensional manifolds are discussed, these actually are open sets in $\mathbb{R}^3$. But a significant part of the considerations depends not on the presence of global coordinates, but on the presence of a global frame of vector fields whose Lie brackets are known. Because of that some of the examples are Lie groups with a given frame of left-invariant vector fields.

In papers 4, 5, 6, 10, 12, 13, 15 a Riemannian metric g is given on an open set in $\mathbb{R}^3$ whose matrix is circulant with first row (A, B, B), $A > B > 0$, as well as a $(1, 1)$-tensor q whose matrix is circulant with first row $(0, 1, 0)$, so q is an isometry and $q^3 = Id$. In thse papers, geometric properties of the pair of tensors (g, q) are studied.

In paper 4, a symmetric non-degenerate bilinear form f with signature $(1, -1, -1)$ and positive determinant is explicitly defined by means of the functions A and B and the tensor q. It is of the form $f(x, y)) = g(x, qy) + g(qx, y)$. For arbitrary smooth functions α and β, the bilinear symmetric form $g_1 = \alpha.g + \beta.f$ is considered. The first theorem of the paper, Theorem 2.1, states that if ∇ and $\dot{\nabla}$ are the Levi-Civita connections of the metrics g and g_1 and if $\nabla q = 0$, then $\dot{\nabla}\ q = 0$ if and only if

$grad\,\alpha = grad\,\beta.S$, where S is the $(1,1)$-tensor whose matrix is circulant with first row $(-1,1,1)$. The problem here is that the form g_1 may degenerate at some points, i.e., g_1 may not be a metric. More precisely, g_1 is a degenerate form at the points x where $\alpha(x) = \beta(x)$ or $\alpha(x) + 2\beta(x) = 0$. Fortunately, after the formulation of the mentioned theorem, in Lemma 2.2, the authors restrict themselves to the case $\alpha(x) > \beta(x) > 0$ for every x, in which case g_1 is positive definite form. Furthermore, in Theorem 2.4, g and g_1 are assumed to be positive definite, and under this assumption, a relation is found between the angles made by a vector $w \in \mathbb{R}^3$ and a non-collinear vector qw computed with respect to the metrics g and g_1.

The form g_1 also appears in paper 5, where it is denoted by $\overline{g}$. In this paper, the conditions $\alpha(x) \neq \beta(x)$ and $\alpha(x) + 2\beta(x) \neq 0$ for every x are imposed on the functions α and β. Under these conditions, the form $\overline{g}$ is non-degenerate, and the authors specify conditions on the functions α and β under which $\overline{g}$ is a positive definite or indefinite metric. In the paper, the authors compute the curvature tensor of the Levi-Civita connection ∇ of the metric g under the condition that the Levi-Civita connection of $\overline{g}$ is flat. And one note. At the second page of the paper, it is claimed that it is obvious that the tensors q, $\widetilde{q}$, f, S are Φ are parallel with respect to ∇. In my opinion, this is not only not obvious, it is not even true in the general case. The parallelism condition of any of these tensors leads to relations between partial derivatives of the functions A and B in the definition of the metric g, and these relations may not hold in the general case. I am making this remark, which does not affect the main result of the paper, only to draw attention to the authors to be more precise in their future work.

The metric g and the $(1,1)$-tensor q in papers 4 and 5 are also studied in paper 6. It is proved that $\nabla q = 0$ exactly when $grad\,A = grad\,B.S$ (notations as above). If x is a tangent vector that is not an eigenvector of q, the angle between x and qx is shown to be between 0 and $\frac{2\pi}{3}$. Let x be a tangent vector such that x, qx, q^2x is an orthonormal basis of the corresponding tangent space. Under the assumption that the tensor q is parallel, it is computed that the sectional curvature of an arbitrary two-dimensional plane $span\{u,v\}$ is proportional to the sectional curvature of the plane $span\{x, qx\}$ with a coefficient of proportionality a rational function of the coefficients of u and v with respect to the basis x, qx, q^2x; this function is found in an explicit form. In the paper, a vector x for which x, qx, q^2x is an orthonormal basis is given in an explicit form involving the functions A and B. If the sectional curvature of $span\{x, qx\}$ were computed for that particular x, the mentioned formula for the sectional curvature of an arbitrary 2-plane would have a larger value.

Article 10 is, in my opinion, the best one and has the most interesting results among the presented papers in the topic we are discussing. Three classes of manifolds are introduced in it: the class $\mathcal{L}_0$ is defined by the condition $\nabla q = 0$, the class $\mathcal{L}_1$ - by the condition $R(x,y,gz,qu) = R(x,y,z,u)$, the class $\mathcal{L}_2$ - by $g(R(qx,qy,qz,qu) = g(R(x,y,z,u)$, where R is the curvature tensor of the metric g and $R(x,y,z,u) = g(R(x,y)z,u)$. These classes can be seen as an analogue of A. Gray's Kähler curvature identities in Hermitian geometry if the tensor q is interpreted as an analogue of the almost complex structure. The class $\mathcal{L}_0$ is an analogue of the class of Kähler manifolds, $\mathcal{L}_1$ - of the Gray class $\mathcal{AH}_1$, $\mathcal{L}_2$ - of the class $\mathcal{AH}_3$. The natural question arises of whether the analogue of the class

$\mathcal{AH}_2$ would be of interest, i.e. whether it is also worth studying the manifolds for which $R(qx, qy, z, u) + R(qx, y, qz, u) + R(qx, y, z, qu) = R(x, y, z, u)$. The class $\mathcal{L}_0$ is characterized in article 6. The classes $\mathcal{L}_1$ and $\mathcal{L}_2$ are characterized by identities for components of the curvature tensor with respect to the standard coordinates of $\mathbb{R}^3$ (which is natural). Much more interesting is the characterization by properties of the Ricci tensor. Let us recall that, since we are considering three-dimensional manifolds, the Ricci tensor completely determines the curvature tensor. I will especially note the result that a manifold is of class $\mathcal{L}_2$ if and only if the Ricci tensor ρ satisfies an identity of the form $\rho = \alpha.g + \beta.f$ where α and β are smooth functions. It follows from this result that the class $\mathcal{L}_2$ contains the Einstein manifolds. On the other hand, as shown in the paper, the Ricci tensor of a manifold of class $\mathcal{L}_1$ is degenerate, therefore the metric of a manifold of this class is not Einstein if it is not Ricci flat. Other curvature properties of the considered classes are also established in the article. Specific examples of manifolds from these classes are also given for which the curvature tensor, the Ricci tensor, and sectional curvatures are computed.

In paper 12, the sectional curvature of 2-planes of the form $span\{x, qx\}$ is computed for manifolds of the classes $\mathcal{L}_1$ and $\mathcal{L}_2$; here x is a tangent vector such that x, qx, q^2x is an orthonormal basis of the corresponding tangent space.

In paper 13, the tensors f and q are denoted by $\widetilde{g}$ and Q, but we will keep the old notations. The tensors $F(x, y, z) = (\nabla_x f)(y, z)$, $\theta(z) = Trace\{x \to F(x, x, z)\}$, $\theta^*(z) = Trace\{x \to F(x, qx, z)\}$ are defined in this paper. If we perceive the tensor f as a symmetric analogue of the second fundamental form of an almost Hermitian manifold, then θ and θ^* can be considered as two versions of the analogue of the Lee form. In paper 13, the components of the tensors F, θ and θ^* are computed. A relation between the Ricci tensors of the metrics g and f is found. A specific example on a Lie group is considered, for which a number of curvature properties are established.

The notations in paper 15 are like those in paper 13, but here we will again stick to the old notations. In paper 15, a relation is found between the Ricci tensors of the metrics g and f. It is shown that a manifold equipped with the pair of tensors (f, q) is of class $\mathcal{L}_0$, respectively, $\mathcal{L}_2$ if and only if the manifold considered with the pair (g, q) is of the same class. It is worth noting the lack of a relation of the $\mathcal{L}_1$ classes defined by (f, q) and (g, q). Curvature properties are established for the pair (f, q), and what is new here is that the metric f is indefinite and considerations related to curvature properties should take into account the fact that there are isotropic vectors and isotropic two-dimensional planes. The paper concludes with concrete computations concerning (f, q) on the Lie group of paper 13.

The subject of papers 4, 5 and 6 is developed in paper 8 in the case when the tensor q is represented by a circulant matrix with first row $(0, -1, 0)$, so q is an isometry and $g^3 = -Id$. Among the results in the paper, I will note the computation of all components of the curvature tensor of the metric g.

The setup and results in paper 7 are similar to those in paper 5, but now the considerations are on open sets of $\mathbb{R}^4$ and q is an isometry satisfying the identity $q^4 = Id$. Of course, the higher dimensions cause some technical complications.

The considerations in paper 7 are continued in paper 9. The main result is the computation of sectional curvatures under the condition $R(qx, qy, qz, qu) = R(x, y, z, u)$. Here I will especially note the interesting example on a Lie group of a Riemannian manifold with metric g and tensor q as well as the computation of the curvature of this manifold.

In paper 11, the pair of tensors (g, q) is considered on two Lie groups defined by a concrete Lie algebras (for the tensor q the notation Q is used in this paper). The components of the curvature tensor are computed. It is found a condition under which the almost product structure $P = q^2$ belongs to the class $\mathcal{W}_1$ introduced by Staikova and Gribachev (1992).

In paper 14, the tensor pair (g, q) is as in papers 7, 9, 11. The geometry of the indefinite metric $\widetilde{g}(u, v) = g(u, q^2 v)$ associated with (g, q) is studied in this paper. Let u be a tangent vector for which u, qu, q^2u, q^3u is a g-orthonormal basis (such a vector exists, as noticed by D. Razpopov). The authors establish that the vector u is space-like $(g(u, u) > 0)$, time-like $(g(u, u) < 0)$ or isotropic $(g(u, u) = 0)$ if and only if the angle between u and q^2u is, respectively, in the interval $(0, \frac{\pi}{2})$, in the interval $(\frac{\pi}{2}, \pi)$, or is 0. They show that in suitable coordinates the "hypersphere" $g(v, v) = a$ is the three dimensional hyperboloid $x'^2 + y'^2 - z'^2 - t'^2 = a$. They also find the form of the section of the "hypersphere" $g(v, v) = a$ with the three dimensional space $span\{u, qu, q^2u\}$. For example, if a $= 0$, this section is a cone or consists of two straight lines. Moreover, they establish the type of "circles", which are the intersections of the "hypersphere" with the two-dimensional spaces $span\{u, qu\}$ and $span\{u, q^2u\}$. The results obtained, though with uncomplicated proofs, make a nice impression with their geometric character.

The candidate's monograph is 104 pages long, with a bibliography of 64 titles (6 pages). The monograph is divided into two chapters. In the first chapter, as in papers 7-9, Riemannian metrics in open subsets of $\mathbb{R}^4$ defined by a circulant matrix and an isometry q with $q^4 = Id$ and ciculant matrix are considered. The contents of the first chapter is almost entirely from the papers on the list of papers for the competition we have already discussed. But Sections 5.1 and 6 are new. In Section 5.1, it is shown that the Ricci tensor ρ of manifolds for which $R(qx, qy, qz, qu) = R(x, y, z, u)$ satisfies an identity of the form $\rho = \alpha.g + \beta.f$, where α and β are smooth functions, similar to the three-dimensional case in paper 10. In section 6 of the first chapter, conditions for the almost product structure $P = q^2$ to belong to the classes introduced by M. Staikova and K. Gribachev are established (in paper 11, this is done only for the class $\mathcal{W}_1$). The results in the second chapter of the monograph are not contained in the papers for the competition. A Riemannian metric g with a skew-circulant matrix is defined in it. A $(1, 1)$-tensor S with such a matrix for which $S^4 = Id$ is also given. Now, we again have two structures on the manifold - one is defined by the pair of tensors (g, S) and the associated neutral metric $g(x, y) = g(x, Sy) + g(Sx, y)$, and the other one is the almost Hermitian structure $(g, J = S^2)$. For the first structure, a number of curvature properties are established with a serious computational work. I will especially note the construction of two examples on Lie groups and the concrete computation of the curvature on them. Similar to paper 14, it is found the geometric form of the "hypersphere" $g(v, v) = a$ and its intersections with two-dimensional and three-dimensional subspaces of the tangent space of a special kind. For the almost Hermitian structure, the condition for it to be locally conformally Kähler is found. I suppose that the candidate will continue the study of this structure to find conditions for the structure to belong to Gray-Hervella classes, to satisfy Kähler curvature identities of Gray, etc., there is a wide field for future work in this direction.

CONCLUSION

The documents and materials presented by Dr. Iva Dokuzova meet the requirements of the Act on the Development of the Academic Staff in the Republic of Bulgaria, the Rules for its implementation and the Rules on the terms and conditions for awarding of academic degrees and occupying academic positions at Plovdiv University.

Based on the comments above, I give a positive assessment of the scientific work of Dr. Iva Rumenova Dokuzova and recommend to the scientific jury to advise the Faculty Council of the Faculty of Mathematics and Informatics, Plovdiv University to appoint Assistant Professor Dr. Iva Rumenova Dokuzova as an Associate Professor in area of higher education 4. Natural Sciences, Mathematics and Informatics, professional field 4.5 Mathematics , scientific speciality Geometry and Topology.

27.03.2024 Reviewer:

(Prof. D.Sc. Johann Davidov)