

R E V I E W

by Prof. Dr. Vejdi Ismailov Hasanov
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on a dissertation for awarding of the educational and scientific degree “Doctor”
in the field of Higher education 4. Natural Sciences, Mathematics and Informatics,

Professional field 4.5 Mathematics

Doctoral program “Mathematical analysis”

Author: Plamena Ivanova Marcheva

Title: Fixed points and convergence of iteration methods for simultaneous approximation of
polynomial zeros

Scientific supervisors:

Prof. DSc Petko D. Proinov – University of Plovdiv “Paisii Hilendarski”

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1. General description of the presented materials

By order No. RD-21-431 / 23.02.2023 of the Rector of the University of Plovdiv "Paisii Hilendarski" (PU), I have been appointed as a member of the scientific jury to ensure a procedure for the defense of a dissertation work on the topic "Fixed points and convergence of iterative methods for simultaneous approximation of polynomial zeros" for the award of the educational and scientific degree "doctor" in: field of higher education 4. Natural sciences, mathematics and informatics, professional direction 4.5. Mathematics, doctoral program “Mathematical analysis”. Author of the dissertation is Plamena Ivanova Marcheva – full-time PhD student at the Department of "Mathematical Analysis" at the Faculty of Mathematics and Informatics of the PU with scientific supervisors Prof. DSc. Petko Proynov and Assoc. prof. Dr. Stoil Ivanov from PU.

The set of materials presented by Plamena Marcheva is in accordance with Art. 36 (1) of the Regulations for development of the academic staff of the PU and includes the following documents:

- a request to the Rector of the PU to open a procedure for the defense of a dissertation work;
- curriculum vitae in European format;

- protocol from the department council related to the discussion of the readiness to open the procedure and preliminary discussion of the dissertation work;
- abstract in Bulgarian and English languages;
- dissertation work;
- a list of scientific publications on the subject of the dissertation;
- copies of scientific publications;
- list of citations;
- declaration of originality and authenticity of the attached documents;
- certificate of fulfillment of the minimum national requirements for the acquisition of the PhD in a professional field 4.5. Mathematics;
- opinion from the academic supervisors on the readiness to defend the presented dissertation work.

Plamena Marcheva has attached three publications on the dissertation work.

2. Brief biographical data

According to the presented autobiography, Plamena Marcheva completed her higher education in 2017 at PU, obtaining a bachelor's degree and a professional qualification of "physicist engineer". During the period of her studies (2015 - 2026) she acquired the additional qualification of "mathematics teacher". In 2018, after studying in a master's program at the PU, she also obtained a master's degree in mathematics.

Plamena Marcheva's professional career began in 2017, as a mathematics teacher at "Chernozets Hrabar" Secondary School, Plovdiv (until June 2019). From September 2022, she is an assistant at PU.

3. Relevance of the topic and appropriateness of the set goals and tasks

The topic of the dissertation is devoted to the task of simultaneously finding the zeros of an algebraic polynomial f of the n -th degree. This task is closely related to the fixed points theory of maps and the convergence study of the corresponding iterative methods. The fixed points theorems are an important tool for proving the existence of a solution to problems in various scientific fields. Regardless of the fact that solving polynomial equations is a special case of the task of solving nonlinear equations, the simultaneous search for all zeros of an algebraic polynomial attracts the attention of quite a few researchers. The first proposed method for simultaneously finding the zeros of an algebraic polynomial is considered to be the Weierstrass method, published in 1891. In the 1960s, this method was rediscovered by other researchers, among whom was the Bulgarian mathematician Kiril Dochev. At that time, Dochev and Burnev proposed an accelerated modification

of the Weierstrass method. Research on the subject remains relevant even today. The increased interest in the subject and numerous studies sometimes lead to the proposal of equivalent iteration formulas.

The aim of the present dissertation is formulated in solving four tasks, which refer to obtaining new improved conditions for local and/or semi-local convergence of a modification of the Weierstrass method and the Dochev and Burnev method, constructing a family of methods of the type of Dochev and Burnev with correction and obtaining conditions for local and semi-local convergence.

4. Knowledge of the problem

The bibliography of the dissertation consists of 108 sources. All literary sources are in English, with the exception of two, which are in Bulgarian and Serbian, respectively. Over 30 publications are from the last 10 years, and it is noticeable that the majority of them are with the participation of Bulgarian researchers. In the introduction to the dissertation work, a detailed literature review on the subject is made, and for this purpose, the sources included in the bibliography are skilfully used. In addition, about 10 monographs on the subject are included in the literature, which is a sign that the topic is current and the author of the present work is well aware of the state of the problem.

5. Characterization and evaluation of the dissertation work

The dissertation consists of 117 pages and includes a list of notations, an introduction, four chapters, a conclusion and a bibliography.

In the introduction, a justification of the developed topic is made, the condition of the problem is presented in detail, and the purpose and tasks of the research are formulated. The conducted research and the obtained results are presented in a summary in each chapter. The presentation of the state of the problem includes a full description of the iterative methods under study, with their respective properties and convergence conditions. The introduction gives a clear idea of the state of the problem and the tasks identified in the dissertation work.

The first chapter has a reference character. Some basic concepts and statements about cone-normalized spaces are presented here, which are a basic tool in proving the convergence of the iterative methods studied in the dissertation. The chapter contains basic definitions and theorems from the convergence theory of iteration methods of the type $x_{k+1} = Tx_k$, $k = 0, 1, \dots$ built by Prof. Proinov for over 10 years. Some of the basic concepts from this theory are: *a function of the initial conditions; quasi-homogeneous function and control function.*

The second chapter is dedicated to the solution of the first task of the research objective set in the dissertation, namely the study of the local and semi-local convergence of the following iteration method:

$$x^{(k+1)} = T(x^{(k)}), \quad k = 0, 1, \dots, \quad (1)$$

where

$$T(x) = (T_1(x), T_2(x), \dots, T_n(x)), \quad T_i(x) = \frac{x_i^2}{x_i + W_i(x)}, \quad W_i(x) = \frac{f(x_i)}{a_0 \prod_{j \neq i} (x_i - x_j)}. \quad (2)$$

This method for simultaneously finding the zeros of an algebraic polynomial f of the n -th order, in the case of prime zeros other than zero, was proposed by Nedzhibov in 2016. It was studied by Nedzhibov in a series of publications and called the inverse Weierstrass method. Here in the thesis the method is called a modification of the Weierstrass method. Convergence is investigated and existing results for local (Theorems 2.1 and 2.2) and semi-local convergence (Theorem 2.3) are improved. The better results are in two directions: first, the neighborhoods of the root-vector ξ for the selection of the initial approximation are expanded by using three types of initial condition functions

$$E(x) = \left\| \frac{x - \xi}{\Delta(\xi)} \right\|_p, \quad E(x) = \left\| \frac{x - \xi}{\Delta(x)} \right\|_p, \quad E_f(x) = \left\| \frac{W(x)}{\Delta(x)} \right\|_p \quad (3)$$

and second, more precise error estimates are obtained. In this chapter, a comparative analysis of the studied method with the classic Weierstrass method is made in theoretical and computational aspects, and the advantages of the Weierstrass method are indicated.

In the third chapter, the method of Dochev and Burnev is studied - iteration formula (1) with components of the iteration function T :

$$T_i(x) = x_i - \frac{f(x_i)}{g'(x_i)} \left(2 - \frac{f'(x_i)}{g'(x_i)} + \frac{1}{2} \frac{f(x_i) g''(x_i)}{g'(x_i) g'(x_i)} \right), \quad g(z) = a_0 \prod_{j=1}^n (z - x_j). \quad (4)$$

Using the technique, through functions of the initial conditions, two results (Theorems 3.1 and 3.2) for local convergence have been obtained, which improve and complement the existing results. New better estimates of the error and the asymptotic constant are obtained.

The last fourth chapter is devoted to a family of methods with accelerated convergence. These methods are also called correction methods, which are obtained by combining two methods, resulting in methods with a higher order of convergence. This technique was applied by Nurein to Ehrlich's method by combining it with Newton's and Weierstrass' methods respectively. In this chapter, the newly constructed family of methods with correction

$$x^{(k+1)} = \mathfrak{S}(x^{(k)}), \quad k = 0, 1, \dots, \quad (5)$$

where $\mathfrak{S}(x) = (\mathfrak{S}_1(x), \mathfrak{S}_2(x), \dots, \mathfrak{S}_n(x))$,

$$\mathfrak{S}_i(x) = x_i - 2\mathcal{W}_i(x) + \mathcal{W}_i^2(x) \left(\frac{f'(x_i)}{f(x_i)} - \sum_{j \neq i} \frac{1}{x_i - \Omega_j(x)} \right), \quad \mathcal{W}_i(x) = \frac{f(x_i)}{a_0 \prod_{j \neq i} (x_i - \Omega_j(x))} \quad (6)$$

is based on the Dochev and Burnev method, which is combined with an arbitrary method with an iteration function $\Omega(x) = (\Omega_1(x), \Omega_2(x), \dots, \Omega_n(x))$. Results are obtained for local (Theorems 4.1 and 4.2) and semi-local (Theorem 4.3) convergence in the general case and for some specific special cases.

6. Contributions and significance of the development for science and practice

The author's contributions in this dissertation work are scientific and scientific-applied in nature. In the conclusion of the paper, the author formulates the contributions in seven points, with which I fully agree. In summary, two types of local convergence conditions are obtained for the studied iteration methods, by which the neighborhood of the root-vector ξ is expanded to select the initial approximation. In a certain sense, this facilitates the selection of the initial approximation, in which the corresponding order of convergence of the considered method is guaranteed. The third type of derived conditions are for semi-local convergence, which are of greater practical importance as they provide computationally verifiable initial conditions. I believe that the tasks set to achieve the goal of the research have been achieved.

7. Evaluation of dissertation publications

The main results of the dissertation are included in three publications. All three publications are co-authored by one or the other supervisor. Because there is no separate protocol for the contributions of co-authors to publications, I consider each one's contribution to be equal. Two of the publications are related to the implementation of Project DN 12/12 of the Scientific Research Fund.

Article 1 of the presented list of publications contains the main results of Chapter 2 and was published in the journal *Symmetry*, which is in the second quartile (Q2) of the Web of Science database. The second is included in a specialized collection of international conference papers (Rhodes, Greece), an edition of the AIP Conference proceedings, which is with SJR. The third publication contains results from Chapter 4 and is included in a collection of reports from a conference for young scientists, published by the Union of Scientists - Plovdiv. The results were reported at three international conferences abroad and one at home. In addition, publication 1 has two citations from foreign authors. According to the minimum national requirements in the

Regulations for the Implementation of the Law on the Development of the Academic Staff in the Republic of Bulgaria, the points required and accumulated by group of indicators are as follows:

A group of metrics	Minimal points	Points accrued
A – dissertation work	50	50
G 7 – scientific publications - with IF from Q2 on WoS - with SJR	30	Total 90 3*20=60 3*10=30
D 11 - citations	-	2*2*4=16

Therefore, the minimum national requirements are exceeded.

I have not noticed the presence of plagiarism, both in the publications and in the dissertation work of Plamena Marcheva.

8. Abstract

The abstract has 32 pages and its content correctly reflects the tasks set in the dissertation, the achieved results and the contribution of the author.

9. Critical remarks and recommendations

I have several critical remarks on the dissertation work and abstract, which are mostly of a technical nature, and others - terminological, but all of them do not detract from the good results achieved. Following are some of the observations:

- In several places there are errors in the typesetting of the text - for example on page 26, line 11.
- In formula (0.22) on page 16, q is not defined up to this point.
- On page 22, line 1, for $\|x\|_\infty$ the “maximal norm ...” is used. This norm is not maximal.
- In Definition 1.5 (iv), the first and third inequalities do not agree.
- On page 29, the last line in the definition of δ , j is redundant.
- Lemma 1.6 is not exactly cited.
- In the second inequality of (1.31), a norm is omitted.
- There is an inaccuracy in the last paragraph of the abstract.

CONCLUSION

The dissertation contains scientific and scientific-applied results, which represent an original contribution to science and meet all the requirements of the Law on the Development of the Academic Staff in the Republic of Bulgaria (LDAS in RBG), the Regulations for the Implementation of the LDAS in RBG and the relevant Regulations of PU. The content of the dissertation work and the publications show that the doctoral student Plamena Ivanova Marcheva has in-depth theoretical knowledge and professional skills in professional field 4.5. Mathematics.

Due to the above, I give my positive assessment for the conducted research, achieved results and contributions, in the presented dissertation work and I propose to the honorable scientific jury to award the educational and scientific degree "doctor" to Plamena Ivanova Marcheva in the field of higher education: 4 Natural sciences, mathematics and informatics, professional field 4.5. Mathematics, PhD program "Mathematical Analysis".

15.03.2023

Reviewer:

(Prof. Dr Vejdi Hasanov)