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Fixed points and convergence of iteration methods for simultaneous approximation of polynomial zeros

ABSTRACT

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The public defense of the dissertation will take place on ...., at .... in the Conference Hall of the New Building of University of Plovdiv ‘Paisii Hilendarski’, Plovdiv.

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The numeration of the theorems and definitions in the abstract coincide with their numerations in the dissertation.

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Relevance and aim of the thesis

Relevance

The fixed point theory and the study of the convergence of high-order iterative methods for approximation of polynomial zeros form two large areas of modern mathematics that are closely interrelated. They are among the most contemporary mathematical problems and have numerous applications in both theoretical and applied research. Midst the most basic applications of the iterative methods are the numerical solving polynomial equations with coefficients in arbitrary normed field and the numerical solving operator equations in Banach spaces.

The problem of obtaining existence and approximation fixed point theorems for mappings in metric spaces dates back to 1922, when Banach published his first work, in which his famous fixed point theorem known as *Banach contraction principle* was presented. Nowadays, this area continues its intensive development and stands at the base of a lot of methods for solving differential, integral and nonlinear operator equations.

It is important to note that the first deep generalizations of the Banach's theorem were obtained by Browder [6] (1968), Boyd and Wong [5] (1969) and Ćirić [11] (1971) while Proinov and Nikolova [83] and Proinov [72] obtained theorems that generalize many fixed point results including the mentioned ones. In 2006 and 2007, using the notations *gauge function of high order* and *iterated contraction mapping*, Proinov [59, 62] obtained theorems that also generalize Banach's theorem. These two works turned out to be a starting point for a new general convergence theory for Picard type iteration developed in 2009-2021 by Proinov [63, 64, 66, 69, 67, 71, 73]. In fact, this theory provides a research method, originated in [59], which can be called *method of the function of initial conditions*.

On the other hand, in the last 60 years the interest of the mathematicians for studying the convergence of iterative methods for approximation of polynomial zeros has strongly increased. In 1994, Sendov, Andreev and Kjurkchiev [94], and recently McNamee [30] (2007) and McNamee and Pan [31] (2013) published monographs dedicated to this topic. Detailed historical survey on iteration methods for polynomial zeros can be found in Pan [43] (1997).

In 1891, Weierstrass [102] offered a qualitatively new approach to approximating zeros of polynomials. He constructed a method for *simultaneous approximation* of all zeros of a complex polynomial (*Weierstrass' method*). The first monograph, entirely dedicated to the simultaneous approximation of polynomial zeros was published in 1989 by PETKOVIĆ [46], though two years earlier Bulgarian mathematician LUBOMIR ILIEV dedicated a chapter of his monograph [20, Ch. 5] to this topic. The next monographs concerning the iteration methods for the simultaneous approximation of polynomial zeros are due to SENDOV, ANDREEV AND KJURKCHIEV [94, Ch. 4] (1994), PETKOVIĆ, HERCEG, AND ILIĆ [52] (1997), KYURKCHIEV [26, Ch. 1, 2, 3, 6] (1998), PETKOV AND KYURKCHIEV [45] (2000), MCNAMEE [30, Ch. 4] (2007), PETKOVIĆ [48] (2008), ILIEV AND KYURKCHIEV [19, Ch. 6] (2010) and CIRA [10] (2012).

Weierstrass method.

As it was mentioned, in 1891, WEIERSTRASS [102] published the first method for the *simultaneous approximation* of all zeros of a complex polynomial f. The *Weierstrass method* is defined in $\mathbb{K}^n$ by the iteration:

$$x^{(k+1)} = x^{(k)} - W(x^{(k)}), \tag{1}$$

where W is defined in $\mathbb{K}^n$ by $W(x) = (W_1(x), \ldots, W_n(x))$ with

$$W_i(x) = \frac{f(x_i)}{a_o \prod_{j \neq i}(x_i - x_j)}\,, \tag{2}$$

where a_0 is the leading coefficient of f.

In the same work WEIERSTRASS [102] proved a semilocal convergence theorem about the method (1) without stating initial conditions that guarantee the convergence of the method but only proving their existence.

As a consequence of this theorem one can obtain the main theorem of Algebra which stands that the field of complex numbers is algebraically closed. Thus, Weierstrass has given the first constructive proof of the main theorem of Algebra. Although, many authors consider Weierstrass' work as an algebraic one, in fact it is an elegant study that lays on the border of Mathematical analysis, Numerical algorithms and Algebra.

In 1913, KÜRSCHAK [25] showed that the arguments of Weierstrass are valid not only in the field of algebraic numbers but in arbitrary valued field. He proved that any algebraic number field can be extended to complete algebraically closed field with an absolute value.

In 1960, DURAND [14], in 1962, Dochev [12] and, in 1966 KERNER [22] rediscovered Weierstrass method and therefore it is often called *Weierstrass-Dochev method*, *Durand-Kerner method* etc. It is important to note that Dochev [12] proved the first local convergence theorem about Weierstrass method. After that, local convergence theorems about Weierstrass method have been proven by Kjurkchiev and Markov [24] (1983), Wang and Zhao [100] (1991), Hopkins, Marshall, Schmidt and Zlobec [17] (1994), Tilli [97] (1998), Han [16] (2000), Niell [40] (2001), Yakoubsohn [104] (2002), Proinov and Petkova [84] (2013) and Proinov and Vasileva [88] (2015).

In 1980, Prešić [57] was the first, after Weierstrass, who published a semilocal convergence theorem about the Weierstrass method. In the next three decades many authors obtained semilocal convergence theorems under different initial conditions: Zheng [107, 108] (1982, 1987), Wang and Zhao [106, 101] (1993, 1995), Petković, Carstensen and Trajković [49] (1995), Petković [47] (1996), Petković and Herceg [50] (1996), Petković, Herceg and Ilić [53] (1998), Batra [3] (1998), Han [16] (2000), Petković and Herceg [51] (2001). In 2006, Proinov [59, 60] published a semilocal convergence theorem that generalizes, improves and complements all previous results in this direction. In 2014, Proinov and Petkova [86], using Weierstrass' ideas, obtained a semilocal convergence theorem about Weierstrass method under a different type of initial conditions. This theorem can be considered as a quantitative version of the Weierstrass' result. A bit later, Proinov [66] has provided a comprehensive study of the local and the semilocal convergence of Weierstrass method, which results improve and complement all existing results of the kind and puts an end to this problem, so far.

Modified Weierstrass method.

In 2016, Nedzhibov [34] published two modifications of Weierstrass method (1). The first of them coincide with Weierstrass method (1) when f has only nonzero roots, i.e., when $f(0) \neq 0$. The second one, that we shall call *modified Weierstrass method*, is defined by the following iteration:

$$x^{(k+1)} = T(x^{(k)}), \qquad k = 0, 1, 2, \ldots, \tag{3}$$

where T is defined in $\mathbb{K}^n$ with $T(x) = (T_1(x), \ldots, T_n(x))$ and

$$T_i(x) = x_i^2 / (x_i + W_i(x)), \tag{4}$$

and $W_i(x)$ is Weierstrass' correction defined by (2).

In recent years, in several articles Nedzhibov [34, 35, 36, 38, 37, 39] has proved local convergence theorems under different initial conditions as well as a semilocal convergence theorem about the modified Weierstrass method (3).

Dochev-Byrnev method.

The second simultaneous method in the literature was presented by Bulgarian mathematicians Dochev and Byrnev [13] in 1964. *Dochev-Byrnev method* is defined by the following fixed point iteration:

$$x^{(k+1)} = T(x^{(k)}) \qquad k = 0, 1, 2, \ldots, \tag{5}$$

where T is defined in $\mathbb{K}^n$ by $T(x) = (T_1(x), \ldots, T_n(x))$ with

$$T_i(x) = x_i - \frac{f(x_i)}{g'(x_i)} \left(2 - \frac{f'(x_i)}{g'(x_i)} + \frac{1}{2} \frac{f(x_i)}{g'(x_i)} \frac{g''(x_i)}{g'(x_i)} \right), \tag{6}$$

and the polynomial g is defined by $g(z) = c_o \prod_{j=1}^{n} (z - x_j)$. In 1972, Prešić [56] rediscovered Dochev-Byrnev method (5) by defining its iteration function $T\colon \mathcal{D} \subset \mathbb{K}^n \to \mathbb{K}^n$ in the following equivalent form:

$$T_i(x) = x_i - W_i(x) \left(1 - \sum_{j \neq i} \frac{W_j(x)}{x_i - x_j} \right), \tag{7}$$

where $W_i(x)$ is the Weierstrass' correction defined by (2).

In 1974, Milovanović [32] gave an elegant derivation of (5) while in 1983, the method (5) was rediscovered again by Tanabe [96] which resulted in it widely known under the name *Tanabe's method* (see e.g. [48] and references therein). In fact, the equivalence of the iteration functions (6) and (7) was proved in only 2016 by Proinov [67, Theorem 4.1].

The local convergence of Dochev-Byrnev method (5) was firstly studied by Semerdzhiev and Pateva [93]. In 1982, Kyurkchiev [23] (see also [94, Theorem 19.1]) proved a theorem that improves the results of the above mentioned authors. In 2011, Toseva, Kyurkchiev and Iliev [98, Theorem A] proved a local convergence result about the method (5) working with the iteration function (7) but according to the equivalence between (7) and (6) this result is in fact a consequence of the mentioned Kyurkchiev's theorem. Very recently, Pavkov, Kabadzhov, Ivanov and Ivanov [44] have obtained local convergence theorems about a family of simultaneous methods that includes, as a special case, Dochev-Byrnev method.

Semilocal convergence theorems about Dochev-Byrnev (Tanabe) method (5) have been proven by Petković, Herceg and Ilić [52, 53] and Ilić and Herceg [18]. In 2016, Proinov [67, Theorem 4.5] proved a semilocal convergence theorem about Dochev-Byrnev method (5) that generalizes and improves all previous such kind of results while Ivanov [21] proved a semilocal convergence theorem about the mentioned family of simultaneous methods that includes Dochev-Byrnev method.

Ehrlich's (Börsch-Supan's) method.

The next simultaneous method in the literature was derived by EHRLICH [15], in 1967, and can be defined by the following fixed point iteration:

$$x^{(k+1)} = F(x^{(k)}) \qquad k = 0, 1, 2, \ldots, \tag{8}$$

where F is defined in $\mathbb{K}^n$ by $F(x) = (F_1(x), \ldots, F_n(x))$, with

$$F_i(x) = x_i - f(x_i)\left(f'(x_i) - f(x_i)\sum_{j\neq i}\frac{1}{x_i - x_j}\right)^{-1}. \tag{9}$$

In 1970, BÖRSCH-SUPAN [4] published the following simultaneous method:

$$x^{(k+1)} = G(x^{(k)}) \qquad k = 0, 1, 2, \ldots, \tag{10}$$

where G is defined in $\mathbb{K}^n$ by $G(x) = (G_1(x), \ldots, G_n(x))$, with

$$G_i(x) = x_i - W_i(x)\left(1 + \sum_{j\neq i}\frac{W_j(x)}{x_i - x_j}\right)^{-1}. \tag{11}$$

In 1982, WERNER [103] proved that the iteration functions (9) and (11) are equivalent (see also Carstensen [7]).

Local and semilocal convergence theorems about Ehrlich's (Börsch-Supan's) method which generalize and improve all previous such results have been obtained by PROINOV [68, 67, 70], IVANOV [21] and PAVKOV, KABADZHOV, IVANOV AND IVANOV [44].

Methods with accelerated convergence.

In 1977, Nourein [41, 42] constructed two fourth-order simultaneous methods based on the methods of Ehrlich and Börsch-Supan. Namely, the first of them is defined by $x^{(k+1)} = \mathcal{F}(x^{(k)})$, where the iteration function $\mathcal{F}$ is defined in $\mathbb{K}^n$ by $\mathcal{F}(x) = (\mathcal{F}_1(x), \ldots, \mathcal{F}_n(x))$, with

$$\mathcal{F}_i(x) = x_i - \frac{f(x_i)}{f'(x_i) - f(x_i)\displaystyle\sum_{j\neq i}\frac{1}{x_i - x_j + f(x_j)/f'(x_j)}}, \tag{12}$$

and the second one is defined by $x^{(k+1)} = \mathcal{G}(x^{(k)})$, where the iteration function $\mathcal{G}$ is defined in $\mathbb{K}^n$ by $\mathcal{G}(x) = (\mathcal{G}_1(x), \ldots, \mathcal{G}_n(x))$ with

$$\mathcal{G}_i(x) = x_i - \frac{W_i(x)}{1 + \sum_{j \neq i} \frac{W_j(x)}{x_i - x_j - W_i(x)}}. \tag{13}$$

Obviously, each of these methods is obtained by combining two methods. The first one is obtained by combining the Ehrlich's iteration function (9) with the famous Newton's one while the second is obtained by combining Börsch-Supan's iteration function (11) with the Weierstrass' iteration function (2). Nowadays, these methods are widely known (see, e.g., [48, Chapter 1]) as:

- Ehrlich's method with Newton's correction;
- Börsch-Supan's method with Weierstrass' correction.

In 2006, Proinov [61] published a semilocal convergence theorem about the method (13) while Proinov and Vasileva [91] proved local and semilocal convergence theorems about the method (12) and thus all previous results about these two methods are now improved.

After the mentioned Nourein's works, many authors (see, e.g., [54, 99, 55, 80, 27, 78]) started to use their ideas in order to construct different methods with accelerated convergence which shall be further called *simultaneous methods with corrections*. In 1987, Wang and Wu [99] were the first who constructed and studied a method with arbitrary correction which however do not accelerated the method's convergence. Very recently, Proinov and Vasileva [92] constructed and studied the local and the semilocal convergence of a family of Ehrlich's type simultaneous methods with arbitrary correction and arbitrary convergence order. In 2023, Proinov and Ivanov [79] constructed and studied a family of Sakurai-Torii-Sugiura type, with arbitrary correction and arbitrary convergence order, for multiple polynomial zeros.

Aim of the dissertation

The aim of the present dissertation is to solve the following problems:

Problem 1. *To investigate the convergence of the modified Weierstrass method* (3) *and thus to obtain local and semilocal convergence theorems that generalize, improve and complement all previous results in this direction.*

Problem 2. *To study the local convergence of Dochev-Byrnev method* (5) *and thus to obtain theorems that generalize, improve and complement all previous results of this kind.*

Problem 3. *To construct a family of simultaneous methods of Dochev-Byrnev type with accelerated convergence. To obtain local and semilocal convergence theorems about the new family as well as about some of its particular members.*

Problem 4. *To conduct some numerical experiments in order to show the applicability of the obtained semilocal convergence results about all of the studied methods.*

A summary of the obtained results

The present dissertation is dedicated to the study of the convergence of the modified Weierstrass method (3) as well as the convergence of a newly constructed family of simultaneous methods of Dochev-Byrnev type with accelerated convergence which includes the classical Dochev-Byrnev method (5).

The dissertation consists of introduction, four chapters, conclusion and references. The conclusion includes: a summary of the results obtained, list of the publications on the dissertation, dissemination of the results and declaration of originality.

The content of the dissertation by chapters and paragraphs is briefly outlined here.

Chapter 1. A general convergence theory for iterative processes in cone normed spaces

This chapter has a referential nature. It is dedicated to a general convergence theory for the convergence of iterative processes in cone normed spaces developed by Proinov [63, 64, 66, 69, 67, 71, 73] during the period from 2009 to 2021.

In **Section 1.1** we present some notations and definitions from the theory of the fields and the theory of the cone normed spaces that are the basis of the present dissertation.

In **Section 1.2**, some important inequalities in $\mathbb{K}^n$ that play a crucial role in the proofs of our results are presented.

Further on, $\mathbb{K}[z]$ denotes the ring of polynomials (of one variable) over the field $\mathbb{K}$.

The vector space $\mathbb{R}^n$ is endowed with the standard coordinate-wise ordering $\preceq$ defined by $x \preceq y \Leftrightarrow x_i \le y_i$ for all $i \in I_n$ and the vector space $\mathbb{K}^n$ is equipped with a p-norm $\|\cdot\|_p$ for some $1 \le p \le \infty$ and with a vector norm $\|\cdot\|$ with values in $\mathbb{R}^n$ defined by

$$\|x\| = (|x_1|, \dots, |x_n|) \quad \text{and} \quad \|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \qquad (1 \le p \le \infty).$$

Also, we define the functions $d\colon \mathbb{K}^n \to \mathbb{R}^n$ and $\delta\colon \mathbb{K}^n \to \mathbb{R}_+$ by

$$d(x) = (d_1(x), \dots, d_n(x)) \;\text{ with }\; d_i(x) = \min_{j \ne i} |x_i - x_j| \;\text{ and }\; \delta(x) = \min_{i \ne j} d_i(x) \tag{14}$$

and for two vectors $x \in \mathbb{K}^n$ and $y \in \mathbb{R}^n$ we use the denotation x/y for the vector in $\mathbb{R}^n$ defined by $x/y = (|x_1|/y_1, \cdots, |x_n|/y_n)$ provided that y has only nonzero components. For a given p $(1 \le p \le \infty)$, we always define q by $1 \le q \le \infty$ with $1/p + 1/q = 1$ and for $n \in \mathbb{N}$, we define the numbers $a = (n-1)^{1/q}$ and $b = 2^{1/q}$.

The main goal of **Section 1.3** is to provide some basic results of the mentioned Proinov's theory for the convergence of the iterative processes of the Picard type $x_{k+1} = Tx_k$, $k = 0, 1, 2, \dots$,where $T : D \subset X \to X$ is an iteration function in cone metric space X over a solid vector space. The main role in this theory is played by the notation *function of initial conditions*.

Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \ge 2$. A vector $\xi \in \mathbb{K}^n$ will be called a *root-vector* of $f \in \mathbb{K}[z]$ if $f(z) = a_0 \prod_{i=1}^{n} (z - \xi_i)$ for all $z \in \mathbb{K}$, where $a_0 \in \mathbb{K}$. Let $\xi \in \mathbb{K}^n$ be a root-vector of f. Examples for functions of initial conditions that are used to prove convergence theorems about simultaneous methods are as follows:

- Function $E\colon \mathbb{K}^n \to \mathbb{R}_+$, defined by

$$E(x) = \left\| \frac{x - \xi}{d(\xi)} \right\|_p . \tag{15}$$

- Function $E\colon \mathcal{D} \to \mathbb{R}_+$, defined by

$$E(x) = \left\| \frac{x - \xi}{d(x)} \right\|_p . \tag{16}$$

- Function $E_f\colon \mathbb{K}^n \to \mathbb{R}_+$, defined by

$$E_f(x) = \left\| \frac{W_f(x)}{d(x)} \right\|_p . \tag{17}$$

All the above formulated functions are used in the present dissertation.

In 2021, Proinov [73] developed his theory by the following definitions that are used in the proofs of the local convergence theorems in Section 2.2 of Chapter 2, in Chapter 3 and Chapter 4.

Definition 1.14 ([73]). The function $F\colon D \subset \mathbb{K}^n \to \mathbb{K}^n$ is said to be an *iteration function of first kind at a point* $\xi \in \mathcal{D}$ if there is a quasi-homogeneous function $\phi\colon J \to \mathbb{R}_+$ of exact degree $m \geq 0$ such that for each $x \in \mathbb{K}^n$ with $E(x) \in J$ the following conditions are satisfied: $x \in D$ and $\| F(x) - \xi \| \preceq \phi(E(x)) \, \|x - \xi\|$, where the function $E\colon \mathbb{K}^n \to \mathbb{R}_+$ is defined by (15). The function ϕ is said to be *control function* of F.

Definition 1.15 ([73]). A function $F\colon D \subset \mathbb{K}^n \to \mathbb{K}^n$ is called *iteration function of second kind at a point* $\xi \in \mathbb{K}^n$ if there exists a nonzero quasi-homogeneous function $\beta\colon J \to \mathbb{R}_+$ such that for each vector $x \in \mathcal{D}$ with $E(x) \in J$ the following hold: $x \in D$ and $\| F(x) - \xi \| \preceq \beta(E(x)) \, \|x - \xi\|$, where the function E is defined by (16). The function β is said to be *control function* of F.

In **Section 1.4**, we provide some theorems of Proinov [69] that gives us a possibility to transform local convergence theorems of the first and second kind into semilocal ones. We have to note that the semilocal convergence theorems are of great practical importance because of their computationally verifiable initial conditions and error estimates.

Chapter 2. New results about a modified Weierstrass method

This chapter consists of five sections and is dedicated to the study of the convergence of the modified Weierstrass method (3).

In **Section 2.1**, the local convergence of the first kind of the modified Weierstrass method (3) is studied. In this section, we present the first main result in the dissertation. This result improves and complements all previous results of the kind about this method:

Theorem 2.1. *Let* $f \in \mathbb{K}[z]$ *be a polynomial of degree* $n \geq 2$ *possessing* n *simple roots in* $\mathbb{K}$ *and such that* $f(0) \neq 0$, $\xi \in \mathbb{K}^n$ *be a root-vector of* f *and* $1 \leq p \leq \infty$. *Suppose* $x^{(0)} \in \mathbb{K}^n$ *is an initial approximation satisfying*

$$E(x^{(0)}) = \left\| \frac{x^{(0)} - \xi}{\Delta(\xi)} \right\|_p < \frac{1}{b} \quad \text{and} \quad \Phi(E(x^{(0)})) < 2, \tag{18}$$

where the function $\Delta\colon \mathbb{K}^n \to \mathbb{R}^n$ *is defined by*

$$\Delta(x) = (\Delta_1(x), \ldots, \Delta_n(x)) \quad \text{and} \quad \Delta_i(x) = \min\{|x_i|, d_i(x)\} \quad (i \in I_n) \tag{19}$$

and the real function Φ *is defined by*

$$\Phi(t) = \frac{1+t}{1-t}\left(1 + \frac{a\,t}{(n-1)(1-bt)}\right)^{n-1}.$$

Then the iteration (3) *is well defined and converges quadratically to* ξ *with error estimates for all* $k \geq 0$

$$\|x^{(k+1)} - \xi\,\| \preceq \lambda^{2^k}\,\|x^{(k)} - \xi\,\| \quad \text{and} \quad \|x^{(k)} - \xi\,\| \preceq \lambda^{2^k-1}\,\|x^{(0)} - \xi\,\|, \tag{20}$$

where $\lambda = \phi(E(x^{(0)}))$ *with* ϕ *defined by*

$$\phi(t) = \frac{\psi(t) - 1 + t}{1 - t - t\,\psi(t)} \quad \text{and} \quad \psi(t) = \left(1 + \frac{a\,t}{(n-1)(1-bt)}\right)^{n-1}. \tag{21}$$

Also, we have the following estimate of the asymptotic error constant:

$$\limsup_{k\to\infty} \frac{\|x^{(k+1)} - \xi\|_p}{\|x^{(k)} - \xi\|_p^2} \leq \frac{a+1}{\widetilde{\Delta}(\xi)}, \tag{22}$$

where $\widetilde{\Delta}(\xi) = \min\{\delta(\xi), \gamma(\xi)\}$, $\delta(\xi) = \min_{i\neq j}|\xi_i - \xi_j|$ *and* $\gamma(\xi) = \min_{i\in I_n}|\xi_i|$.

In **Section 2.2**, the local convergence of the second kind of the modified Weierstrass method (3) is studied.

Before stating our main result, we define the functions γ and β by

$$\gamma(t) = \left(1 + \frac{a\,t}{n-1}\right)^{n-1} \quad \text{and} \quad \beta(t) = \frac{\gamma(t) - 1 + t\,\gamma(t)}{1 - t\,\gamma(t)}. \tag{23}$$

The next theorem is our main result in this section. It improves a result of Nedzhibov [39].

Theorem 2.2. *Let* $f \in \mathbb{K}[z]$ *be a polynomial of degree* $n \geq 2$ *which has* n *simple zeros in* $\mathbb{K}$ *and* ξ *be a root-vector of* f. *Suppose* $x^{(0)} \in \mathbb{K}^n$ *is an initial approximation with pairwise distinct nonzero components satisfying the initial condition*

$$E(x^{(0)}) = \left\|\frac{x^{(0)} - \xi}{\Delta(x^{(0)})}\right\|_p < \eta \quad \text{and} \quad \Phi(E(x^{(0)})) \leq 2, \tag{24}$$

where the real function Φ *is defined by* $\Phi(t) = (1 + (2+b)t)\,\gamma(t)$ *with* γ *defined by* (23) *and* η *is the unique solution of the equation* $t\,\gamma(t) = 1$ *in the interval* $[0, \infty)$. *Then the iteration* (3) *is well defined and converges Q-quadratically to* ξ *with the following error estimates for all* $k \geq 0$:

$$\|x^{(k+1)} - \xi\,\| \preceq \theta\,\lambda^{2^k}\,\|x^{(k)} - \xi\,\| \quad \text{and} \quad \|x^{(k)} - \xi\,\| \preceq \theta^k\,\lambda^{2^k-1}\,\|x^{(0)} - \xi\,\|, \tag{25}$$

where $\lambda = \phi(E(x^{(0)}))$, $\theta = \psi(E(x^{(0)}))$, $\phi = \beta/\psi$ *and* $\psi(t) = 1 - bt(1 + \beta(t))$. *Besides, the estimate of the asymptotic error constant* (22) *holds.*

In **Section 2.3**, a semilocal convergence theorem about the method (3) is obtained.

Theorem 2.3. *Let* $f \in \mathbb{K}[z]$ *be a polynomial of degree* $n \geq 2$, *and let* $x^{(0)} \in \mathbb{K}^n$ *be an initial approximation with pairwise distinct nonzero components satisfying*

$$E_f(x^{(0)}) = \left\| \frac{W(x^{(0)})}{\Delta(x^{(0)})} \right\|_p < \frac{R(1 + (b-1)R)}{(1 + bR)(1 + (a + b - 1)R)}\,, \tag{26}$$

where R *is the number*

$$R = \frac{\sqrt[n-1]{h} - 1}{b\,(\sqrt[n-1]{h} - 1) + a/(n-1)}$$

and the number h *is given by*

$$h = \frac{3b - a - 1 + \sqrt{(3b - a - 1)^2 + 8(b+1)(a + 1 - b)}}{2(b+1)}\,.$$

Then f *has only simple zeros in* $\mathbb{K}$ *and the iteration* (3) *is well defined and converges quadratically to a root-vector* ξ *of* f.

In **Section 2.4** several numerical experiments are conducted to show the applicability of Theorem 2.3.

In **Section 2.5** we present a theoretical and a numerical comparison between the classical Weierstrass method and the modified Weierstrass method.

Chapter 3. New results about Dochev-Burnev method

Third chapter is dedicated to the study of the local convergence of Dochev-Burnev method (5).

In **Section 3.1**, a local convergence theorem of the first kind is proved. Before stating our theorem, we define the real functions ϕ and μ, by

$$\phi(t) = (\mu(t) - 1)^2 + \frac{at^2\,\mu(t)^2}{(1-t)(1-bt)} \quad \text{and} \quad \mu(t) = \left(1 + \frac{a\,t}{(n-1)(1-b\,t)}\right)^{n-1}. \tag{27}$$

Theorem 3.1. *Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ which has n simple zeros in $\mathbb{K}$, $\xi \in \mathbb{K}^n$ be a root-vector of f and $1 \leq p \leq \infty$. Suppose $x^{(0)} \in \mathbb{K}^n$ is an initial approximation satisfying*

$$E(x^{(0)}) < 1/b \quad \textit{and} \quad \Phi(E(x^{(0)})) < 2, \tag{28}$$

where E is defined by (15) and Φ is defined by

$$\Phi(t) = \mu(t)\left(1 + \frac{at^2}{(1-t)(1-bt)}\right)$$

with μ defined by (27). Then Dochev-Byrnev iteration (5) is well defined and converges Q-cubically to ξ with the following error estimates for all $k \geq 0$:

$$\|x^{(k+1)} - \xi\| \preceq \lambda^{3^k} \|x^{(k)} - \xi\| \quad \textit{and} \quad \|x^{(k)} - \xi\| \preceq \lambda^{(3^k-1)/2} \|x^{(0)} - \xi\|, \tag{29}$$

where $\lambda = \phi(E(x^{(0)}))$ and ϕ is defined by (27). Also, we have the following estimate of the asymptotic error constant:

$$\limsup_{k\to\infty} \frac{\|x^{(k+1)} - \xi\|_p}{\|x^{(k)} - \xi\|_p^3} \leq \frac{a}{\delta(\xi)^2}\,, \tag{30}$$

where δ is defined by (14).

In **Section 3.2**, we prove the second main result in this chapter, which is a local convergence theorem of the second kind about Dochev-Byrnev method (5).

For the purposes of the theorem, we define the function β by

$$\beta(t) = (\mu(t) - 1)^2 + \frac{a\,t^2\,\mu(t)^2}{1-t}\,, \text{ where } \mu(t) = \left(1 + \frac{a\,t}{n-1}\right)^{n-1} \tag{31}$$

and the functions Ψ and ψ as follows:

$$\Psi(t) = 1 - bt - \beta(t)(1+bt) \quad \text{and} \quad \psi(t) = 1 - bt(1+\beta(t)). \tag{32}$$

Theorem 3.2. *Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ which has n simple zeros in $\mathbb{K}$, $\xi \in \mathbb{K}^n$ be a root-vector of f and $1 \leq p \leq \infty$. Suppose $x^{(0)} \in \mathbb{K}^n$ is an initial approximation satisfying*

$$E(x^{(0)}) < 1 \quad \textit{and} \quad \Psi(E(x^{(0)})) \geq 0, \tag{33}$$

where E is defined by (15) and Ψ is defined by (32). Then Dochev-Byrnev iteration (5) is well defined and converges Q-cubically to ξ with the following error estimates for all $k \geq 0$:

$$\|x^{(k+1)} - \xi\| \preceq \theta\lambda^{3^k} \|x^{(k)} - \xi\| \quad \textit{and} \quad \|x^{(k)} - \xi\| \preceq \theta^k\lambda^{(3^k-1)/2} \|x^{(0)} - \xi\|, \tag{34}$$

where $\lambda = \phi(E(x^{(0)}))$, $\theta = \psi(E(x^{(0)}))$, $\phi = \beta/\psi$ and ψ is defined by (32).

Chapter 4. A new family of methods with accelerated convergence

This chapter consists of six sections and is dedicated to the study of the convergence of a newly constructed family of simultaneous methods of Dochev-Byrnev type with accelerated convergence.

In **Section 4.1**, we first construct a new family of simultaneous methods of Dochev-Byrnev type with accelerated convergence which is called *Dochev-Byrnev method with correction* and then we investigate its local convergence of the first kind.

Let $\Omega\colon \mathscr{D} \subset \mathbb{K}^n \to \mathbb{K}^n$ is an arbitrary iteration function then we define the following family of simultaneous methods:

$$x^{(k+1)} = \mathcal{T}(x^{(k)}), \qquad k = 0, 1, 2, \dots, \tag{35}$$

where $\mathcal{T}$ is defined by $\mathcal{T}(x) = (\mathcal{T}_1(x), \dots, \mathcal{T}_n(x))$ and

$$\mathcal{T}_i(x) = x_i - 2\mathcal{W}_i(x) + \mathcal{W}_i(x)^2 \left(\frac{f'(x_i)}{f(x_i)} - \sum_{j \neq i} \frac{1}{x_i - \Omega_j(x)} \right),$$

and $\mathcal{W}_i(x)$ is defined by $\mathcal{W}_i(x) = f(x_i)/(a_0 \prod_{j \neq i} (x_i - \Omega_j(x)))$.

For an arbitrary quasi-homogeneous function $\omega\colon J \to \mathbb{R}_+$ of exact degree $m \geq 0$ and an integer $n \geq 2$, we define the function $\gamma\colon J \to \mathbb{R}_+$ as follows

$$\gamma(t) = t\,(1 + \omega(t)^q)^{1/q}\,. \tag{36}$$

Using the function γ defined by (36), we define the function ϕ as follows:

$$\phi(t) = (\mu(t) - 1)^2 + \frac{a\,\mu(t)^2 \omega(t)\,t^2}{(1-t)(1-\gamma(t))} \quad \text{with} \quad \mu(t) = \left(1 + \frac{a\,t\,\omega(t)}{(n-1)(1-\gamma(t))}\right)^{n-1}. \tag{37}$$

The next theorem is the main result in this section.

Theorem 4.1. *Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ which has n simple zeros in $\mathbb{K}$, $\xi \in \mathbb{K}^n$ be a root-vector of f, $1 \leq p \leq \infty$ and let $\Omega\colon \mathscr{D} \subset \mathbb{K}^n \to \mathbb{K}^n$ be an iteration function of first kind at ξ with control function $\omega\colon J \to \mathbb{R}_+$ of exact degree $m \geq 0$. Suppose $x^{(0)} \in \mathbb{K}^n$ is an initial approximation satisfying*

$$E(x^{(0)}) \in J, \quad \gamma(E(x^{(0)})) < 1 \quad \textit{and} \quad \Phi(E(x^{(0)})) < 2, \tag{38}$$

where E is defined by (15), *γ is defined by* (36) *and Φ is defined as follows:*

$$\Phi(t) = \mu(t)\left(1 + \frac{a\,\omega(t)\,t^2}{(1-t)(1-\gamma(t))}\right) \tag{39}$$

with μ defined by (37)*. Then the iteration* (35) *is well defined and converges to ξ with Q-order $r = m + 3$ and with the following error estimates for all $k \geq 0$:*

$$\|x^{(k+1)} - \xi\,\| \preceq \lambda^{r^k}\,\|x^{(k)} - \xi\,\| \quad \textit{and} \quad \|x^{(k)} - \xi\,\| \preceq \lambda^{\frac{r^k-1}{r-1}}\,\|x^{(0)} - \xi\,\|, \tag{40}$$

where $\lambda = \phi(E(x^0))$ and ϕ is defined by (37)*. Also, we have the following estimate of the asymptotic error constant:*

$$\limsup_{k\to\infty} \frac{\|x^{(k+1)} - \xi\|_p}{\|x^{(k)} - \xi\|_p^r} \leq \frac{a}{\delta(\xi)^{\,r-1}} \lim_{t\to 0^+} \frac{\omega(t)}{t^m}\,, \tag{41}$$

where δ is defined by (14)*.*

In **Section 4.2**, a local convergence theorem of the second kind about the method (35) is obtained.

Before stating our theorem, for an arbitrary quasi-homogeneous function $\omega\colon J \to \mathbb{R}_+$ of exact degree $m \geq 0$ and an integer $n \geq 2$, we define the function $\gamma\colon J \to \mathbb{R}_+$ as follows:

$$\gamma(t) = \begin{cases} t\,(1+\omega(t)), & \textit{if } \Omega \textit{ is not identity function,} \\ 0, & \textit{if } \Omega \textit{ is identity function.} \end{cases} \tag{42}$$

Using the function γ, we define the function β as follows:

$$\beta(t) = (\mu(t) - 1)^2 + \frac{a\,\mu(t)^2\omega(t)\,t^2}{(1-t)(1-\gamma(t))}\,, \tag{43}$$

where μ is defined by (37).

The next theorem is the main result in this section.

Theorem 4.2. *Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ which has n simple zeros in $\mathbb{K}$, $\xi \in \mathbb{K}^n$ be a root-vector of f, and let $\Omega\colon \mathscr{D} \subset \mathbb{K}^n \to \mathbb{K}^n$ be an iteration function of second kind at ξ with control function $\omega\colon J \to \mathbb{R}_+$ of exact degree $m \geq 0$. Suppose $x^{(0)} \in \mathbb{K}^n$ is an initial approximation satisfying*

$$E(x^{(0)}) \in J \cap [0,1), \quad \gamma(E(x^{(0)})) < 1 \quad \textit{and} \quad \Psi(E(x^{(0)})) \geq 0, \tag{44}$$

where E is defined by (16) *and* Ψ *is defined by* (32) *with* β *defined by* (43)*. Then the iteration* (35) *is well defined and converges to* ξ *with Q-order* $r = m+3$ *and with the following error estimates for all* $k \geq 0$*:*

$$\|x^{(k+1)} - \xi\| \preceq \theta\, \lambda^{r^k} \|x^{(k)} - \xi\| \text{ and } \|x^{(k)} - \xi\| \preceq \theta^k\, \lambda^{\frac{r^k-1}{r-1}} \|x^{(0)} - \xi\|, \quad (45)$$

where $\lambda = \phi(E(x^{(0)}))$*,* $\theta = \psi(E(x^{(0)}))$*,* ψ *is defined by* (32) *with* β *defined by* (43) *and* $\phi = \beta/\psi$*.*

In **Section 4.3**, as consequences of Theorem 4.1 are Theorem 4.2, we obtain local convergence theorems of the first and second kind about the following particular members of our new family (35):

(i) **Dochev-Byrnev method (DB)**, if $\Omega(x) = x$.

(ii) **Dochev-Byrnev method with Weierstrass' correction (DBW)**, if Ω is Weierstrass' iteration function defined by (2).

(iii) **Dochev-Byrnev method with Newton's correction (DBN)**, if Ω is Newton's iteration function $\Omega_i(x) = x_i - f(x_i)/f'(x_i)$.

(iv) **Dochev-Byrnev method with Ehrlich's correction (DBE)**, if Ω is Ehrlich's iteration function defined by (9).

(v) **Dochev-Byrnev method with Halley's correction (DBH)**, if Ω is Halley's iteration function in $\mathbb{K}^n$ defined by

$$\Omega_i(x) = x_i - \frac{f(x_i)}{f'(x_i)} \left(1 - \frac{1}{2}\frac{f(x_i)}{f'(x_i)}\frac{f''(x_i)}{f'(x_i)}\right)^{-1}.$$

In **Section 4.4**, we get the following semilocal convergence theorem about Dochev-Byrnev method with correction (35):

Theorem 4.3. *Suppose* $f \in \mathbb{K}[z]$ *is a polynomial of degree* $n \geq 2$ *with only simple zeros in* $\mathbb{K}$ *and let* $\Omega\colon \mathscr{D} \subset \mathbb{K}^n \to \mathbb{K}^n$ *be an iteration function of second kind with control function* $\omega\colon J \to \mathbb{R}_+$ *of exact degree* $m \geq 0$*. Suppose* $x^{(0)} \in \mathbb{K}^n$ *is an initial guess with pairwise distinct components satisfying the following initial conditions:* $E_f(x^{(0)}) < 1/(1+\sqrt{a})^2$*,* $h(E_f(x^{(0)})) \in J$*,* $\gamma(h(E_f(x^{(0)}))) < 1$ *and* $\Psi(h(E_f(x^{(0)}))) \geq 0$*, where* E_f*,* γ *and* Ψ *are defined by* (17)*,* (42) *and* (32) *with* β *defined by* (43) *and* h *defined by*

$$h(t) = 2t\Big/\Big(1 - (a-1)t + \sqrt{(1-(a-1)t)^2 - 4t}\Big). \quad (46)$$

Then f *has only simple zeros in* $\mathbb{K}$ *and the iteration* (35) *is well defined and converges to a root-vector* ξ *of* f *with Q-order* $r = m + 2$*.*

In **Section 4.5**, as a consequence of Theorem 4.3, we get semilocal convergence theorems about the above defined particular members of the family (35).

In **Section 4.6**, two numerical examples which show the applicability of Theorem 4.3 and emphasize the behavior of the above defined particular members of the family (35) are conducted.

Conclusion

Summary of the obtained results

The main contributions in the present dissertation are:

1. Two local convergence theorems (Theorem 2.1 and Theorem 2.2) with apriori and aposteriori error estimates and with an estimate of the asymptotic error constant of the modified Weierstrass method (3) are obtained. These theorems improve and complement all previous results of the kind about this method.

2. A semilocal convergence theorem (Theorem 2.3) which improves and complements all previous such results about the modified Weierstrass method is obtained. This theorem is of great practical importance because of its computationally verifiable initial conditions.

3. Based on the obtained results, theoretical and numerical comparisons between the modified Weierstrass method and the classical Weierstrass method is provided in Section 2.5. Numerical examples are presented to show the applicability of the semilocal theorem (Section 2.4).

4. Two local convergence theorems (Theorem 3.1 and Theorem 3.2) under two different kinds of initial conditions, error estimates and estimate of the asymptotic error constant of Dochev-Byrnev method are obtained. The first of them generalize, improve and complement all previous such results and the second one is the first of the kind in the mathematical literature.

5. A new family of simultaneous methods of Dochev-Byrnev type with accelerated convergence (*Dochev-Byrnev method with correction*) is constructed.

6. Two local convergence theorems (Theorem 4.1 and Theorem 4.2) about Dochev-Byrnev method with correction are obtained. As consequences, local convergence theorems about four particular members of the family, obtained by using the famous iteration functions of Weierstrass, Newton, Ehrlich and Halley are proven. It is proven that the convergence order of the first two methods is four, while the convergence order of the others is five.

7. A semilocal convergence theorem (Theorem 4.1) about Dochev-Byrnev method with correction and some theorems about the mentioned four members are obtained. Two numerical examples which show the applicability of the theorems and emphasize the behavior of the particular members are conducted with given polynomials and initial approximations.

List of publications

The main results on the dissertation are included in the following three papers:

1. Plamena I. Marcheva, Stoil I. Ivanov, Convergence analysis of a modified Weierstrass method for the simultaneous determination of polynomial zeros, Symmetry, 12(9): Article No. 1408, 19 pages, 2020, **Q2 (IF 2.713)**. https://doi.org/10.3390/sym12091408

2. Plamena I. Marcheva, Stoil I. Ivanov, On the semilocal convergence of a modified Weierstrass method for the simultaneous computation of polynomial zeros, AIP Conf. Proc., 2425: Article No. 420012, 4 pages, 2022. **(SJR 0.177)**. https://doi.org/10.1063/5.0082007.

3. Petko D. Proinov, Plamena I. Marcheva, A new family of Dochev-Byrnev type iterative methods with accelerated convergence, Sci. Works Union Sci. Bulg., 23(Series B) page 93–97, 2022. **ISSN: 2534–9376**.

The connections between contributions, aims, problems and their place in the description of the dissertation and in the publications are presented in the following table:

Contribution	Aim	Problem	Section	Publication
1	1	1	2.1, 2.2	1
2	1	1	2.3	1, 2
3	1	1, 4	2.4, 2.5	1, 2
4	1	2	3.1, 3.2	3
5	1	3	4.1	3
6	1	3	4.1, 4.2, 4.3	3
7	1	3, 4	4.4, 4.5	3

Dissemination of the results

A) Talks at conferences and seminars

- Plamena I. Marcheva and Stoil I. Ivanov, Local convergence of modified Weierstrass method for the simultaneous determination of polynomial zeros, 9th International Eurasian conference on mathematical sciences and applications (IECMSA 2020), Skopje, North Macedonia, August 25-28, 2020. http://www.iecmsa.org/program/ (page 8)
- Plamena I. Marcheva and Stoil I. Ivanov, On the semilocal convergence of a modified Weierstrass method for the simultaneous computation of polynomial zeros, 18th International conference of numerical analysis and applied mathematics, Rhodes, Greece, September 25-28, 2020. http://history.icnaam.org/icnaam_2020/ICNAAM
- Plamena I. Marcheva, On the convergence of a Weierstrass type method for the simultaneous approximation of polynomial zeros, 1-st Western Balkan Conference in Mathematics and Applications, (WBCMA2021), Pristina, Kosovo. http://fwbcma2021.ilirias.com/
- Plamena I. Marcheva, A new family of Dochev-Byrnev type iterative methods with accelerated convergence, IX-th International conference of young scientists, Plovdiv, Bulgaria, July 14th, 2022.

B) Scientific projects

- 'Study of high-order iterative methods for approximation of polynomial zeros and fixed points of quasi-contraction maps in metric spaces.', National Science Fund of the Bulgarian Ministry of Education and Science under Grant DN 12/12, 2017–2020.

Declaration of originality

from **Plamena Ivanova Marcheva**,
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In connection with the procedure for acquiring the educational and scientific degree 'Doctor of Philosophy (PhD)' at University of Plovdiv 'Paisii Hilendarski' and the defense of the presented dissertation, I declare:

The results and contributions of the conducted research presented in my dissertation entitled 'Fixed points and convergence of iteration methods for simultaneous approximation of polynomial zeros' are original and are not borrowed from research and publications without my participation.

13.02.2023. DECLARANT:
Plovdiv /Plamena Ivanova Marcheva/

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