

OPINION

by **prof. Vejdi Ismailov Hasanov, PhD**

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on a dissertation for the award of the scientific degree "Doctor of Sciences" (DSc)
in the field of Higher education 4. Natural Sciences, Mathematics and Informatics,
Professional field 4.5 Mathematics (Mathematical analysis)

Author: Prof. Dr. Boyan Georgiev Zlatanov – University of Plovdiv “Paisii Hilendarski”

Title: Applications of coupled fixed points and coupled best proximity points

1. General presentation of the procedure and the author of the dissertation

By order No. RD-21-1333 / 18.07.2022 of the Rector of Plovdiv University "Paisii Hilendarski" (PU), I have been appointed as a member of the scientific jury to ensure a procedure for the defense of a dissertation work on the topic "Applications of coupled fixed points and coupled best proximity points" for the award of the scientific degree "Doctor of Sciences" in the field of higher education 4. Natural sciences, mathematics and informatics, professional direction 4.5. Mathematics (Mathematical analysis). Author of the dissertation is Prof. Dr. Boyan Georgiev Zlatanov – Department of "Mathematical Analysis" at the Faculty of Mathematics and Informatics (FMI) of the PU.

The set of materials presented by Prof. B. Zlatanov is in accordance with Art. 45 (4) of the Regulations for development of the academic staff of the PU, includes the following documents: request to the Rector of the PU to disclose the procedure for the defense of a dissertation work; CV in European format; copy of diploma for the educational and scientific degree "doctor"; protocols from the department council, regarding the preliminary discussion of the dissertation work; dissertation work; abstract in Bulgarian and English languages; a list of scientific publications on the subject of the dissertation and a reference to the points they carry according to the minimum national requirements; copies of the publications from the above list; a list of all scientific works with information about their application in the author's academic development procedures; lists of citations: of publications on the dissertation work; of all publications and citations used in the procedure with the relevant points; declaration of originality and authenticity of the attached documents; certificate of compliance with the minimal national requirements; diploma for ESD "Doctor".

The candidate has attached 18 publications – 12 are in scientific journals and 6 in conference proceedings.

Prof. B. Zlatanov graduated with a degree in "Mathematics" from the FMI of the Sofia University "St. Kliment Ohridski" in 1996. In 2001, after studying at the PU and defending his thesis, he obtained a PhD in the scientific specialty "Mathematical Analysis". The working career of Prof. Zlatanov is entirely related to PU, since 1999 until now. In April 2008 and October 2019,

he held the academic positions "Associate Professor" and "Professor", respectively. In addition, since December 2015, he has been the vice dean for research and project activities at FMI.

2. Relevance of the topic

The topic of the dissertation is on the applications of theorems for coupled fixed points and coupled best proximity points. Fixed point theorems are an important tool for proving the existence of a solution to problems in various scientific fields. From the fundamental theorems: Brauer's (1911), its generalizations – Schauder's theorem (1930), Kakutani (1941); Banach contraction principle (1922), followed by the theorem of Browder (1965), Ran and Reurings (2004), etc., until now, research on this topic has not lost its relevance. In recent years, there has been increased interest in coupled fixed points defined on partially ordered spaces.

3. Knowledge the problem

From the review of the literature used and the exposition of the dissertation work, it can be concluded that the author is well acquainted with the development and state of the problem.

4. Characterization and evaluation of the dissertation work and contributions

The dissertation is of 314 pages, containing: preface, introduction, 5 chapters, conclusion with contributions and literature from 134 sources. Each chapter has a concluding paragraph.

In the introduction, some commonly used notations are described, the basic concepts and known results on the subject are defined, some of which are summarized or further developed in the following chapters.

The first chapter is devoted to coupled fixed points in partially ordered metric spaces. In the first paragraph, a generalization of Ekeland's variational principle for continuous maps with mixed monotone property is proposed (Theorem 1.1). In the following paragraphs, using the variational principle, the existence of coupled fixed points of maps of different types with a mixed monotone property are proved. The results of the first three paragraphs are published in publication No. 6 of the attached list, the following in the fourth paragraph were reported at the International Conference "Application of Mathematics in Engineering and Economics", Sozopol, 2019 and included in publication No. 7, and those of fifth paragraph were reported at the MATTECH 2022 Scientific Conference with international participation.

The second chapter is devoted to the problem of finding coupled best proximity points and error estimation. The task of finding the point/s of best approximation arises in cyclic maps where no fixed points exist. In the general case, in the presence of a fixed point, the best proximity points coincide with the fixed points. Considering the question of coupled fixed points, here the notion of best proximity points is extended to coupled best proximity points. The main result in the first paragraph contains error estimates for best proximity points (Theorem 2.1), which was published in publication No. 2. Using the technique of Theorem 2.1. in the next two paragraphs, error estimates for coupled best proximity points for cyclic contractive maps are obtained. The results of the second paragraph are published in publication No. 3, and of the third paragraph in publications No. 10 and 11. In the fourth paragraph, coupled fixed points and coupled best proximity points for p-cyclic contractive maps are investigated. Theorems for the existence and uniqueness of a coupled fixed points of a p-cyclic maps (Theorem 2.13) and a

coupled best proximity points for a p -cyclic contractive maps (Theorem 2.14) are proved, respectively. The results of the fourth paragraph are published in publication No. 12.

In the third chapter, fixed points, coupled fixed points, best proximity points, and coupled best proximity points in modular functional spaces are studied. The main results are published in Publications Nos. 1, 4 and 5.

In the fourth chapter, applications of the coupled fixed points and the coupled best proximity points are discussed in the study of the existence of equilibrium in duopoly markets. For a more adequate modeling of the relations of two players (producers) in duopoly markets, semi-cyclic maps are used instead of the cyclic maps discussed in the previous chapters. In the first two paragraphs, definitions are given for the basic concepts related to semi-cyclic maps and a theorem for the existence of a unique ordered pair of fixed points of a semi-cyclic ordered pair of maps (Theorem 4.1). In the third paragraph, examples of different market participants and an application of Theorem 4.1 are discussed. In the fourth paragraph, results are obtained for coupled fixed points for semi-cyclic maps of the Hardy–Rogers type, which are published in publication No. 16. More significant results in the next two paragraphs are Theorem 4.4 and Theorem 4.5, respectively, which are published in publications with Nos. 15 and 13. The seventh paragraph is devoted to coupled best proximity points for semi-cyclic maps.

In the last fifth chapter, tripled fixed points and tripled best proximity points are considered. In this chapter, the ideas of ordered pairs of ordered maps pairs are extended to ordered pairs of ordered maps triples and coupled fixed points and coupled best proximity points, respectively to tripled fixed points and tripled best proximity points. The obtained results are applied to the study of market equilibrium with three dominant participants. The main results are published in publications Nos. 9 and 14 and in press in No. 17.

In the dissertation most of the results are supported by practical examples. The results and contributions of the author have a scientific and scientific-applied character.

I have not noticed the presence of plagiarism in the works of Prof. B. Zlatanov.

5. Evaluation of the publications and personal contribution of the dissertation student

The publications on the dissertation are 18, of which 12 are in scientific journals, 2 in proceedings of the International Conference "Application of Mathematics in Engineering and Economics", which are referenced in Scopus, 2 are in proceedings of the Scientific Conference MATTECH with international participation and 2 are under print in proceedings. A strong impression is made by 4 publications that are in journals with a high impact factor and fall into the Q1 quartile. Another 4 are in journals in Q2. In addition, 6 of the publications are independent, of which 4 are in Q1 or Q2 quartile journals, 3 are with one co-author, and the rest are co-authored by two or more. I assume that the dissertation's contribution to the general works is equal.

The publications on the dissertation work are cited in 41 publications by reference of the author, of which only two are not in refereed editions. The entire scientific output of Prof. B. Zlatanov has been cited in at least 248 publications. All this is a clear indicator that the results in the dissertation work and beyond have received a wide response among the mathematical community.

Undoubtedly, the minimum national requirements for the science degree "Doctor of science" have been met.

6. Abstract

The abstract is 64 pages long and correctly reflects the content of the dissertation work and the contributions of the author.

7. Personal impressions

I know Prof. B. Zlatanov as a responsive and fair colleague. The presented dissertation leaves me with the impression of a precise mathematician with in-depth knowledge of the researched topic.

CONCLUSION

The dissertation contains scientific and scientific-applied results, which represent an original contribution to science and meet all the requirements of the Law on the Development of the Academic Staff in the Republic of Bulgaria (LDASRB), the Regulations for the Implementation of the LDASRB and the relevant Regulations of PU "Paisii Hilendarski".

Due to the above, I confidently give my positive assessment of the achieved results and contributions in the presented dissertation work and I propose to the honorable scientific jury to award the scientific degree "Doctor of Sciences" to Prof. Dr. Boyan Georgiev Zlatanov in the field of higher education 4 Natural sciences, mathematics and informatics, professional direction 4.5. Mathematics (Mathematical Analysis).

02.09.2022

Opinion prepared by:

(Prof. Vejdi Hasanov, PhD)