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Applications of Coupled Fixed Points and Coupled Best Proximity Points

SUMMARY

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The dissertation "Application of Coupled Fixed Points and Coupled Best Proximity Points" contains 314 pages. It consists of a preface, introduction, 5 chapters, a conclusion, a bibliography. The bibliography contains 133 sources. The list of author's publications on the dissertation consists of 18 items.

The public defence of the thesis will take place on .... at .... in the meeting hall of the New Building of Plovdiv University "Paisii Hilendarski", Plovdiv, 236 Bulgaria Blvd.

The documents are available to those interested in the Dean's Secretary of the Faculty of Mathematics and Informatics, New building of Plovdiv University "Paisii Hilendarski", office 330, every working day from 8:30 to 17:00.

Contents

General characteristics of the thesis

In the presented dissertation, generalizations of Banach 's fixed point theorem related to coupled fixed points, coupled best proximity points and their applications are considered. Generalizations of Ekeland's variational principle are considered, which are related to sets generated by maps with the mixed monotone property. A technique for proving results about existence of coupled fixed points for maps with the mixed monotone property, using the generalizations of the variational principle, is proposed. The results for the existence and uniqueness of coupled best proximity points are enriched by finding the error estimates, when using sequences of successive iterations. It has been proved for coupled fixed points and coupled best proximity points (x, y) that they must satisfy $x = y$, provided that the classical model from [26] is used. A generalization of the concepts of coupled fixed points and coupled best proximity points, ordered pair of cyclic maps is proposed, which allows the ordered pair (x, y) to consist of two different points. An approach is proposed to reduce systems of equations to a problem for coupled fixed points or coupled best proximity points. The possibilities for finding exact solutions of systems of equations with the help of the enriched theory of coupled fixed points and coupled best proximity points are illustrated. The concepts of coupled fixed points and coupled best proximity points in modular functional spaces are introduced. Possibilities for solving systems of equations with the help of cyclic maps in modular functional space, which is generated by the system of equations, are illustrated. A new class of maps is defined, which is different from both cyclic and non-cyclic maps. This class is called semi-cyclic maps. This class arises naturally in the study of market equilibrium in duopoly markets. Conditions for the existence and uniqueness of coupled fixed points and coupled best proximity points for semi-cyclic maps have been found. Models of duopoly markets have been constructed with the help of semi-cyclic maps, which significantly enriches the classical theory of duopoly markets. The obtained results are illustrated with different models. The ideas for generalizing of coupled fixed points and coupled best proximity points have been further developed for tripled fixed points and tripled best proximity points, as well as semi-cyclical maps of three variables, which naturally arise when modeling markets dominated by three participants.

INTRODUCTION

Fixed point theorems, initiated by Banach's Contraction Principle [5] has proved to be a powerful tool in nonlinear analysis.

Fixed point theory of course entails the search for a combination of conditions on a set X and a mapping $T : X \to X$ which, in turn, assures that T leaves at least one point of X fixed, i.e. $\xi = T(\xi)$ for some $\xi \in X$. Since its publication [5] there is large number of applications and generalizations.

There are two main directions in the generalizations. The first one is to alter the underlying space X and the second one is to alter the contractive type condition.

Notations, used in the thesis

We will denote the set of all natural numbers by $\mathbb{N}$ and the set of all real numbers by $\mathbb{R}$. With the capital letters A, B, C, X, Y, Z we will denote sets of arbitrary structure. We will denote with the small letters x, y, z, w, u, v, t the elements of the considered sets. We will denote by ρ the metric defined in a metric space (X, ρ).

We will denote also by ρ the function modular, that defines the modular function space L_ρ. As far as the metric function $\rho(\cdot,\cdot)$ depends on two variables and the function modular $\rho(\cdot)$ depends on one variable, and they are considered in different chapters, there will be no misunderstanding.

By $X \times Y$ we will consider the Cartesian product, i.e. $u = (x, y) \in X \times X$.

A distance bwtween two subsets $A, B \subset X$, provided that (X, ρ) is a metric space, is defined by the function $\operatorname{dist}(A, B) = \inf\{\rho(x, y) : x \in A, y \in B\}$.

Partially ordered metric spaces and maps with the mixed monotone property

Following [11, 26], let X be a set and let $\preceq$ be a partial order in X, then $(X, \preceq)$ is called a partially ordered set.

Definition 1. *([11, 26]) Let $(X, \preceq)$ be a partially ordered set and let $F : X \times X \to X$. The function F is said to have the mixed monotone property if*

$$\text{for any } x_1, x_2, y \in X, \quad \text{such that } x_1 \preceq x_2 \text{ there holds } F(x_1, y) \preceq F(x_2, y)$$

and

$$\text{for any } y_1, y_2, x \in X, \quad \text{such that } y_1 \preceq y_2 \text{ there holds } F(x, y_1) \succeq F(x, y_2).$$

Following [11, 26] let $(X, \rho, \preceq)$ be a partially ordered complete metric space. We endow the product space $X \times X$ with the following partial order $(u, v) \preceq (x, y)$, provided that $x \succeq u$ and $y \preceq v$ holds simultaneously and with the following metric

$$d((x, y), (u, v)) = \rho(x, u) + \rho(y, v)$$

for $(x, y), (u, v) \in X \times X$.

Coupled fixed points in partially ordered metric spaces

One direction for generalization of fixed points is the notion of coupled fixed points [26], where maps with the mixed monotone property in partially ordered by a cone Banach spaces are investigated. Later this idea was developed for maps with the mixed monotone property in partially ordered metric spaces [11].

Definition 2. *([11, 26]) Let X be a set and let $F : X \times X \to X$. An ordered pair $(x, y) \in X \times X$ is called coupled fixed point of F if $x = F(x, y)$ and $y = F(y, x)$.*

The authors in [11] have refined the technique from [26] in order to generalize the results from [26]. They have presented an easier to apply technique for the investigation of the existence and uniqueness of coupled fixed points, which is widely used today.

Ekeland's variational principle

Ekeland proved a variational principle in [18]. In a series of articles [19, 20] he enriches the results. Later he presented a more concise proof [21], which technique we will use. In the same article [21], various applications of the variational principle in different fields of mathematics are presented.

Ekeland's variation principle has many generalizations and applications in different fields of Mathematics [13, 16].

Uniformly convex Banach spaces

The best proximity results need norm-structure of the underlying space X.

When we investigate a Banach space $(X, \|\cdot\|)$, we will always consider the distance between the elements to be generated by the norm $\|\cdot\|$, i.e. $\rho(x, y) = \|x - y\|$. We will denote the unit sphere and the unit ball of a Banach space $(X, \|\cdot\|)$ by S_X and B_X respectively.

The assumption that the Banach space $(X, \|\cdot\|)$ is uniformly convex plays a crucial role in the investigation of best proximity points.

Definition 3. *([15]) Let $(X, \|\cdot\|)$ be a Banach space. For every $\varepsilon \in (0, 2]$ we define the modulus of convexity of $\|\cdot\|$ by $\delta_{\|\cdot\|}(\varepsilon) = \inf \left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in B_X, \|x - y\| \geq \varepsilon\right\}$. The norm is called uniformly convex if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. The space $(X, \|\cdot\|)$ is then called a uniformly convex Banach space.*

Lemma 1. *([22]) Let A be a nonempty closed, convex subset, and B be a nonempty closed subset of a uniformly convex Banach space. Let $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be sequences in A and $\{y_n\}_{n=1}^{\infty}$ be a sequence in B satisfying:*

(a) $\lim_{n\to\infty} \|x_n - y_n\| = \operatorname{dist}(A, B)$

(b) $\lim_{n\to\infty} \|z_n - y_n\| = \operatorname{dist}(A, B)$

then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.

Lemma 2. *([22]) Let A be a nonempty closed, convex subset, and B be a nonempty closed subset of a uniformly convex Banach space. Let $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be sequences in A and $\{y_n\}_{n=1}^{\infty}$ be a sequence in B satisfying:*

(a) $\lim_{n\to\infty} \|z_n - y_n\| = \operatorname{dist}(A, B)$

(b) for every $\varepsilon > 0$ there is $N_0 \in \mathbb{N}$, such that for all $m > n \geq N_0$, $\|x_m - y_n\| \leq \operatorname{dist}(A, B) + \varepsilon$,

then for every $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$, such that for all $m > n > N_1$, holds $\|x_m - z_n\| \leq \varepsilon$.

Cyclic maps, fixed points and best proximity points

Definition 4. *([36]) Let A and B be nonempty subsets of a metric space (X, ρ). The map $T : A \bigcup B \to A \bigcup B$ is called a cyclic map if $T(A) \subseteq B$ and $T(B) \subseteq A$.*

For simplicity of the notations or to fit some of the formulas into the text field, when no misunderstanding can appear, we will denote $\operatorname{dist}(A, B)$ with d.

Definition 5. *([22]) Let A and B be nonempty subsets of a metric space (X, ρ) and $T : A\bigcup B \to A\bigcup B$ be a cyclic map. A point $\xi \in A$ is called a best proximity point of the cyclic map T in A if $\rho(\xi, T\xi) = \mathrm{dist}(A, B)$.*

Definition 6. *([22]) Let A and B be nonempty subsets of a metric space (X, ρ). The map $T : A\bigcup B \to A\bigcup B$ is called a cyclic contraction map if T is a cyclic map and for some $k \in (0, 1)$ there holds the inequality $\rho(Tx, Ty) \le k\rho(x, y) + (1 - k)\mathrm{dist}(A, B)$ for any $x \in A$, $y \in B$.*

The concept of coupled fixed point theorem is introduced in [26].

Definition 7. *([11, 26]) Let A be nonempty subset of a metric space (X, ρ), $F : A\times A \to A$. An ordered pair $(x, y) \in A \times A$ is said to be a coupled fixed point of F in A if $x = F(x, y)$ and $y = F(y, x)$.*

Definition 8. *([49]) Let A and B be nonempty subsets of X. The ordered pair of maps (F, f), $F : A \times A \to B$ and $f : B \times B \to A$ is called an ordered pair of cyclic maps.*

The concept of coupled best proximity points theorem is introduced in [49].

Definition 9. *([49]) Let A and B be nonempty subsets of a metric space X, $F : A\times A \to B$. An ordered pair $(x, y) \in A \times A$ is called a coupled best proximity point of F in A if*

$$\rho(x, F(x, y)) = \rho(y, F(y, x)) = \mathrm{dist}(A, B).$$

It is easy to see that if $A = B$ in Definition 8, then a coupled best proximity point reduces to a coupled fixed point.

Definition 10. *([27, 49]) Let A and B be nonempty subsets of a metric space X, $F : A\times A \to B$ and $G : B\times B \to A$. The ordered pair (F, G) is said to be a cyclic contraction of type two, if there exist non-negative numbers α, β, such that $\alpha + \beta < 1$ and there holds the inequality*

$$\rho(F(x, y), G(u, v)) \le \alpha\rho(x, u) + \beta\rho(y, v) + (1 - (\alpha + \beta))d(A, B)$$

for all $(x, y) \in A \times A$ and $(u, v) \in B \times B$.

Definition 11. *([49])Let $A, B \subset X$. Let $F : A \times A \to B$ and $G : B \times B \to A$. For any pair $(x, y) \in A \times A$ we define the sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ by $x_0 = x$, $y_0 = y$ and*

$$\begin{array}{ll} x_{2n+1} = F(x_{2n}, y_{2n}), & y_{2n+1} = F(y_{2n}, x_{2n}) \\ x_{2n+2} = G(x_{2n+1}, y_{2n+1}), & y_{2n+2} = G(y_{2n+1}, x_{2n+1}) \end{array}$$

for all $n \ge 0$.

Best proximity points of p–cyclic contractions

A generalization of the notion of best proximity points for p sets is obtained in [33] for p–cyclic contraction maps and in [32] for p–cyclic Meir–Keeler contraction maps. The ideas from [32, 33] is developed for p–cyclic maps of Kanan type in [45].

Let $\{A_i\}_{i=1}^{p}$ be nonempty subsets of a metric space (X, d). Following [32, 33], a well known agreement just to simplify the notations is $A_{p+i} = A_i$ for any $i \in \mathbb{N}$. A map $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ is called a p–cyclic map if $T(A_i) \subseteq A_{i+1}$ for every $i = 1, 2, \dots, p$. A point $\xi \in A_i$ is called a best proximity point of T in A_i if $d(\xi, T\xi) = \mathrm{dist}(A_i, A_{i+1})$, provided that T be a p–cyclic map.

Error estimates for fixed points, obtained by a sequence of successive iterations

There are many problems about fixed points and best proximity points that are not easy to be solved or could not be solved exactly. One of the advantages of Banach fixed point Theorem is the error estimates of the successive iterations.

That is why an estimation of the error when an iterative process is used is of interest. An extensive study about approximations of fixed points can be found in [6].

Unfortunately error estimates for best proximity points were missing. Th first result in this direction is obtained in [52]. The advantage of the presented results [52] is that a direct stop criteria of the iteration process is presented, when an exact solution is not possible to be found. The second benefit of the presented technique is that it widens the classes of equations for which an approximation of the solution can be found with sequences of successive iterations.

Modular function spaces

Besides the idea of defining a norm and considering a Banach space, another direction of generalization of the Banach Contraction Principle is based on considering an abstractly given functional defined on a linear space, which controls the growth of the members of the space. This functional is usually called modular and it defines a modular space. The theory of modular spaces was initiated by Nakano [43] in connection with the theory of ordered spaces, which was further generalized by Musielak and Orlicz [42]. Modular function spaces are subclass of the modular spaces. The study of the geometry of modular function spaces was initiated by Kozlowski [38, 39, 40].

Since the theory of modular functional spaces is not as well known as that of metric, partially ordered metric or Banach spaces, we will try to systematize the basic definitions, concepts and results for modular functional spaces in more detail. We will follow the survey paper [41].

Let Ω be a nonempty set and Σ be a nontrivial σ–algebra of subsets of Ω. Let $\mathcal{P}$ be a δ–ring of subsets of Ω, such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Let us assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$, such that $\Omega = \cup K_n$. By $\mathcal{E}$ we denote the linear space of all simple functions with supports from $\mathcal{P}$. By $\mathcal{M}_\infty$ we will denote the space of all extended measurable functions, i.e. all functions $f : \Omega \to [-\infty, \infty]$ such that there exists a sequence $\{g_n\} \subset \mathcal{E}$, $|g_n| \leq |f|$ and $g_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$. By $\mathbf{1}_A$ we denote the characteristic function of the set A.

Definition 12. *Let $\rho : \mathcal{M}_\infty \to [0, \infty]$ be a nontrivial convex and even function. We say that ρ is a regular convex function pseudo-modular if:*

(i) $\rho(0) = 0$*;*

(ii) *ρ is monotone, i.e., $|f(\omega)| \leq |g(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f) \leq \rho(g)$, where $f, g \in \mathcal{M}_\infty$;*

(iii) *ρ is orthogonaly subadditive, i.e., $\rho(f\mathbf{1}_{A\cup B}) \leq \rho(f\mathbf{1}_A) + \rho(f\mathbf{1}_B)$, where $A, B \in \Sigma$ such that $A \cap B \neq \emptyset$, $f \in \mathcal{M}_\infty$;*

(iv) *ρ has the Fatou property, i.e., $|f_n(\omega)| \uparrow |f(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f_n) \uparrow \rho(f)$, where $f \in \mathcal{M}_\infty$;*

(v) *ρ is order continuous in $\mathcal{E}$, i.e., $g_n \in \mathcal{E}$ and $|g_n(\omega)| \downarrow 0$ implies $\rho(g_n) \downarrow 0$.*

Similarly as in the case of measure spaces, we say that a set $A \in \Sigma$ is ρ–null if $\rho(g\mathbf{1}_A) = 0$ for every $g \in \mathcal{E}$. We say that a property holds ρ–almost everywhere, if the exceptional set is ρ–null. As usual we identify any pair of measurable sets, whose symmetric difference is ρ–null as well as any pair of measurable functions differing only on a ρ–null set. With this in mind we define $\mathcal{M}(\Omega, \sigma, \mathcal{P}, \rho) = \{f \in \mathcal{M}_\infty; |f(\omega)| < \infty\rho \ - \ a.e.\}$, where each $f \in \mathcal{M}(\Omega, \sigma, \mathcal{P}, \rho)$ is actually an equivalence class of functions equal ρ a.e. rather than an individual function. Where no confusion exists we will write $\mathcal{M}$ instead of $\mathcal{M}(\Omega, \sigma, \mathcal{P}, \rho)$.

Definition 13. *Let ρ be a regular convex function pseudo-modular.*

(1) *We say that $\rho(0)$ is a regular convex function semi-modular if $\rho(\alpha f) = 0$ for every $\alpha > 0$ implies $f = 0$ ρ–a.e.;*

(2) *We say that ρ is a regular convex function modular if $\rho(f) = 0$ implies $f = 0$ ρ–a.e.*

The class of all nonzero regular convex function modular defined on Ω will be denoted by $\mathfrak{R}$.

Let us denote $\rho(f, E) = \rho(f 1_E)$ for $f \in \mathcal{M}$, $E \in \Sigma$. It is easy to prove that $\rho(f, E)$ is a function pseudo-modular in the sense of Definition 12 [38].

Definition 14. *Let ρ be a convex function modular.*

(a) *A modular function space is the vector space $L_\rho(\Omega, \Sigma)$, or briefly L_ρ, defined by*

$$L_\rho = \{f \in \mathcal{M} : \rho(\lambda f) \to 0 \text{ as } \lambda \to 0\}$$

(b) *The following formula defines a norm in L_ρ (frequently called Luxemburg norm):*

$$\|f\|_\rho = \inf \left\{ \alpha > 0 : \rho\left(\frac{f}{\alpha}\right) \le 1 \right\}.$$

In the present study, when we formulate something in terms of a norm in a modular functional space, we will understand the Luxembourg norm $\|\cdot\|_\rho$, generated by ρ.

In this way, Lebesgue, Orlicz, Musielak–Orlicz spaces are examples of modular function spaces.

Geometry of modular function spaces

Generalization of convexity properties for Banach spaces are investigated for modular function spaces in [35]. As demonstrated in [41], one concept of uniform convexity for Banach spaces generates several different types of uniform convexity in modular function spaces. This is due primarily to the fact that in general the modular function is not homogeneous.

Definition 15. *Let $\rho \in \mathfrak{R}$ and $i \in \{1, 2\}$. Let $r > 0$, $\varepsilon > 0$. Define*

$$D_i(r, \varepsilon) = \{(f, g) : f, g \in L_\rho, \rho(f) \le r, \rho(g) \le r, \rho\left(\frac{f-g}{i}\right) \ge \varepsilon r\}.$$

Let $\delta_i(r, \varepsilon) = \inf\left\{1 - \frac{1}{r}\rho\left(\frac{f+g}{2}\right) : (f, g) \in D_i(r, \varepsilon)\right\} > 0$ if $D_i(r, \varepsilon) \neq \emptyset$ and $\delta_i(r, \varepsilon) = 1$ if $D_i(r, \varepsilon) = \emptyset$.

(i) We say that ρ satisfies (UCi) if for any $r > 0$, $\varepsilon > 0$ there holds the inequality $\delta_i(r, s) > 0$.

(ii) We say that ρ satisfies (UUCi) if for every $s \geq 0$, $\varepsilon > 0$ there exists $\eta_i(s, \varepsilon) > 0$, depending on s and ε such that $\delta_i(r, s) > \eta_i(s, \varepsilon) > 0$ for $r > s$.

If ρ is (UC1) we obtain that the inequality $\rho\left(\frac{x+y}{2}\right) \leq r\left(1 - \delta_1(r, \varepsilon)\right)$ holds for every $\rho(x), \rho(y) \leq r$ and $\rho(x - y) \geq r\varepsilon$.

Proposition 1. *The following conditions characterize relationship between the notions, that are defined in Definition 15*

(1) (UUCi) implies (UCi) for $i \in 1, 2$;

(2) $\delta_1(r, \varepsilon) \leq \delta_2(r, \varepsilon)$;

(3) (UC1) implies (UC2);

(4) (UUC1) implies (UUC2);

(5) If $\rho \in \mathfrak{R}$, then (UUC1) and (UUC2) are equivalent;

(6) If ρ is homogeneous (e.g. is a norm) then all conditions (UC1), (UC2), (UUC1) and (UUC2) are equivalent.

Orlicz function spaces

We recall that M is called an Orlicz function, provided M is even, convex, continuous non-decreasing in $[0, \infty)$ function with $M(0) = 0$, $M(t) > 0$ for any $t \neq 0$. Let M be an Orlicz function and let (Ω, Σ, μ) be a measure space. Let us consider the space $L^0(\Omega)$ consisting of all measurable real–valued functions on Ω and define for every $f \in L^0(\Omega)$ the Orlicz function modular $\widetilde{M}(f) = \int_\Omega M(f(t))d\mu(t)$.

Definition 16. *The Orlicz space $L_M(\Omega, \Sigma, \mu)$ is the space of all classes of equivalent μ–measurable functions $f : \Omega \to \mathbb{R}$ over the measure space (Ω, Σ, μ) such that $\widetilde{M}(\lambda f) \to 0$ as $\lambda \to 0$ or equivalently $\widetilde{M}\left(\frac{f}{\lambda}\right) < \infty$ for some $\lambda > 0$.*

The function $\widetilde{M}$ is a regular convex function modular and it is called Orlicz function modular.

We say that M satisfies the Δ_2–condition if there exist constants $C, t_0 > 0$, such that $M(2t) \leq CM(t)$ for any $t \geq t_0$. It is easy to observe that if M satisfies the Δ_2–condition, then the Orlicz function modular $\widetilde{M}$ has the Δ_2 property.

If we restrict to the Orlicz space $L_M(0, 1)$, then the Orlicz function modular is defined by $\widetilde{M}(f) = \displaystyle\int_0^1 M(f(s))d\mu(s)$. We will denote the corresponding modular function space by $L_{\widetilde{M}}(0, 1)$. When $M = |t|^p$ we will denote $L_{\widetilde{M}}(0, 1)$ by $L_{\widetilde{p}}(0, 1)$.

Definition 17. *([34], p. 81) A function φ is said to be very convex if for any $\varepsilon > 0$ and any u_0 there exists $\delta > 0$ such that $\varphi\left(\frac{u-v}{2}\right) \geq \frac{\varepsilon}{2}(\varphi(u) + \varphi(v)) \geq \varepsilon\varphi(u_0)$ implies $\varphi\left(\frac{u-v}{2}\right) \leq \frac{1-\delta}{2}(\varphi(u) + \varphi(v))$.*

Definition 18. *[31] A function φ is said to be uniformly convex on whole $\mathbb{R}$ if for any $a \in (0,1)$ there exists $\delta(a) \in (0,1)$ such that $\varphi\left(\frac{u+au}{2}\right) \leq (1-\delta(a))\frac{\varphi(u)+\varphi(au))}{2}$.*

If φ is uniformly convex then φ is very convex [35]. Examples of uniformly convex Orlicz functions are $M(t) = |t|^p$, $p > 1$.

The uniform convexity of the Orlicz function implies (UC1) [31]. It is known [35, 41] that for Orlicz spaces over a finite, atom-less measure space the Orlicz modular $\widetilde{M}$ is (UC2) if and only if M is very convex.

It can be proved ([34], p. 116) that in Orlicz spaces over a finite atom-less measure the uniform continuity of the Orlicz function modular is equivalent to the Δ_2–condition.

Fixed points for multi–valued maps

Following the "Banach Contraction Principle", Nadler introduced the concept of multi–valued contractions [30]. Let Y be a set. We denote by 2^Y the set of all subsets of Y. Let X and Y be sets. A map $f : X \to 2^Y$ is called a multi–valued map, i.e. the map f associates with any $x \in X$ a subset $f(x)$ of Y. The set $f(x) \subseteq Y$ is called an image of x under f. We will denote the set valued maps by $f : X \rightrightarrows Y$.

Definition 19. *([30]) A point $x \in X$ is said to be a fixed point of the multi–valued mapping $F : X \rightrightarrows X$ if $x \in F(x)$.*

Definition 20. *([48]) A point $(x;y) \in X \times X$ is said to be a coupled fixed point of the multi–valued mapping $F : X \times X \rightrightarrows X$ if $x \in F(x,y)$ and $y \in F(y,x)$.*

Tripled fixed points and tripled best proximity points

A lot of results in modeling real world processes in applied mathematics lead to the problems, where T depends on more than two variables, e.g. $T : X \times X \times X \to X$. The theory of tripled fixed points and tripled best proximity points [2, 9, 47] is a generalization of coupled fixed points introduced in [11]. This idea have been further generalized for quadrupled fixed points [44] ans n–order (n–tuple) fixed points [47].

Following [4, 9, 12] we will give one possible definition for a tripled fixed point.

Definition 21. *([9]) An element $(x,y,z) \in X \times X \times X$ is called a tripled fixed point of $F : X \times X \times X \to X$ if there hold $x = F(x,y,z)$, $y = F(y,x,y)$ and $z = F(z,y,x)$.*

The notion of maps with the mixed monotone property is generalized for maps of three variables $F : X \times X \times X \to X$ in [9].

Definition 22. *([9]) Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \times X \to X$. We say that F has the mixed monotone property if $F(x,y,z)$ is monotone non-decreasing in x and z, and is monotone non-increasing in y, that is for any $x,y,z \in X$,*

$$\text{for } \ x_1, x_2 \in X, \quad x_1 \preceq x_2 \ \text{ there holds } \ F(x_1,y,z) \preceq F(x_2,y,z),$$

$$\text{for } \ y_1, y_2 \in X, \quad y_1 \preceq y_2 \ \text{ there holds } \ F(x,y_1,z) \succeq F(x,y_22,z)$$

and

$$\text{for } \ z_1, z_2 \in X, z_1 \preceq z_2 \ \text{ there holds } \ F(x,y,z_1) \preceq F(x,y,z_2).$$

It is proposed in [47] a different approach in generalizing of coupled fixed points.

Definition 23. *([47]) Let X be a non-empty set and $F : X^N \to X$ be a given mapping, $(N \geq 2)$. An element $(x_1, x_2, \dots, x_N) \in X^N$ is said to be a fixed point of N–order of the mapping F if*

$$\begin{array}{rcl} F(x_1, x_2, \dots, x_N) & = & x_1, \\ F(x_2, x_3, \dots, x_N, x_1) & = & x_2, \\ \dots & \dots & \dots \\ F(x_N, x_1, \dots, x_{N-1}) & = & x_N. \end{array}$$

Definition 24. *([47]) Let (X, ρ) be a metric space and $F : X^N \to X$ be a given mapping. Let M be a non-empty subset of X^{2N}. We say that M is F–invariant subset of X^{2N} if for all $x_1, x_2, \dots, x_{2N} \in X$, we have*

$$(x_1, x_2, \dots, x_{2N}) \in M \quad \textit{if and only if} \quad \left\{ \begin{array}{rcl} (x_2, x_3, \dots, x_{2N}, x_1) & \in & M \\ (x_3, x_4, \dots, x_{2N}, x_1, x_2) & \in & M \\ \dots & \dots & \dots \\ (x_{2N}, x_1, \dots, x_{2N-1}) & \in & M \end{array} \right.$$

and $(F(x_1, x_2, \dots, x_N), \dots, F(x_N, x_1, \dots, x_{N-1}), F(x_{N+1}, x_{N+2}, \dots, x_{2N}), \dots,$ $F(x_{2N}, x_{N+1}, \dots, x_{2N-1})) \in M$, provided that $(x_1, x_2, \dots, x_{2N}) \in M$.

We see that Definition 24 actually replaces the partial order.

Oligopoly markets

Let's agree that market participants are divided into two types. These are the ones who want to sell their product and we will call them producers (manufacturer) or firms (companies) and those who buy the product we will call them buyers or consumers. We assume that shops, resellers, exchanges are intermediate in the sale of a product from producer to consumer.

Let's first look at the duopoly market [24, 50], where two companies compete for the same customers and strive to meet market demand with total output of $Z = x + y$, where x and y are the production quantities of producers one and two, respectively. The market price is set as $P(Z) = P(x+y)$, which is the inverse of the demand function. Let each manufacturer have a cost function of $c_1(x)$ and $c_2(y)$, respectively. We assume that both participants have rational behavior. The payoff functions for both participants are respectively $\Pi_1(x, y) = xP(x + y) - c_1(x)$ and $\Pi_2(x, y) = yP(x + y) - c_2(y)$. Due to the assumption of rational behavior of producers and the fact that everyone accepts the quantities of goods produced by their competitor as fixed, the maximization of the payoff of each of the participants can be recorded in the form $\max\{\Pi_1(x, y) : x,$ assuming that y is fixed$\}$ and $\max\{\Pi_2(x, y) : y,$ assuming that x is fixed$\}$.

Provided that functions P and c_i, $i = 1, 2$ are differentiable, we get the equations

$$(1) \qquad \left| \begin{array}{l} \frac{\partial \Pi_1(x,y)}{\partial x} = P(x + y) + xP'(x + y) - c_1'(x) = 0 \\ \frac{\partial \Pi_2(x,y)}{\partial y} = P(x + y) + yP'(x + y) - c_2'(y) = 0. \end{array} \right.$$

The solution of (1) presents the equilibrium pair of production in the duopoly market, provided that second order conditions are satisfied [24, 50]. The second order conditions are

$$\left| \begin{array}{l} \frac{\partial^2 \Pi_1(x,y)}{\partial x^2} < 0 \\ \frac{\partial^2 \Pi_2(x,y)}{\partial y^2} < 0. \end{array} \right. \tag{2}$$

Chapter I
COUPLED FIXED POINTS IN PARTIALLY ORDERED METRIC SPACES

We will present some possible generalizations of known results [11, 26] about coupled fixed points in partially ordered complete metric spaces. We will start first with a generalization of Ekeland's variational principle, which generalization we will apply in the proofs of the results about existence of coupled fixed points in partially ordered metric spaces for maps with the mixed monotone property.

A generalization of Ekeland's variational principle for maps with the mixed monotone property

Just to fit some of the formulas in the text field we will use in this chapter the following notation $\overline{u} = \big(u^{(2)}, u^{(1)}\big)$ for $u = \big(u^{(1)}, u^{(2)}\big) \in X \times X$, where $u \in X \times X$. .

Theorem 1. *Let $(X, \rho, \preceq)$ be a partially ordered complete metric space, $(X \times X, d, \preceq)$ and $F : X \times X \to X$ be a continuous map with the mixed monotone property. Let*

$$V \times V = \{x = (x^{(1)}, x^{(2)}) \in X \times X : x^{(1)} \preceq F(x) \text{ and } x^{(2)} \succeq F(\overline{x})\} \neq \emptyset.$$

Let $T : X \times X \to \mathbb{R} \cup \{+\infty\}$ be a proper, l.s.c, bounded from below function. Let $\varepsilon > 0$ be arbitrary and let $u_0 \in V \times V$ be an ordered pair such that the inequality $T(u_0) \leq \inf_{V \times V} T(v) + \varepsilon$ holds. Then there exists an ordered pair $x \in V \times V$, such that

(i) $T(x) \leq T(u_0)$;

(ii) $d(x, u_0) \leq 1$;

(iii) For every $w \in V \times V$ different from $x \in V \times V$ holds the inequality

$$T(w) > T(x) - \varepsilon d(w, x).$$

Coupled fixed points results for maps with the mixed monotone property obtained with the help of a variational technique

Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$. Following [26], for any $(\xi_0, \eta_0) \in X \times X$ we will consider the sequence $\{\xi_n, \eta_n\}_{n=0}^{\infty}$, defined by $\xi_n = F(\xi_{n-1}, \eta_{n-1})$ and $\eta_n = F(\eta_{n-1}, \xi_{n-1})$ for $n \in \mathbb{N}$.

We will give an alternative proof of ([7], Theorem 3) for the existence of coupled fixed points using the variational principle from the previous section.

Theorem 2. *Let* $(X,\rho,\preceq)$ *be a partially ordered complete metric space,* $(X\times X,d,\preceq)$ *and* $F: X\times X\to X$ *be a continuous map with the mixed monotone property. Let there exists* $\alpha\in[0,1)$*, so that the inequality*

$$\rho(F(x,y),F(u,v))+\rho(F(y,x),F(v,u))\le\alpha\rho(x,u)+\alpha\rho(y,v)$$

holds for all $x\succeq u$ *and* $y\preceq v$*. If there exists at least one ordered pair* (x,y)*, such that* $x\preceq F(x,y)$ *and* $y\succeq F(y,x)$*, then there exists a coupled fixed points* (x,y) *of* F*.*

If in addition every pair of elements in $X\times X$ *has an lower or an upper bound, then the coupled fixed point is unique.*

Theorem 2 slightly generalizes the result from [11].

It was proved in [14] the existence and uniqueness of coupled fixed points for Kannan type maps in metric space. We present a generalization in the context of mixed monotone maps in partially ordered metric spaces.

Theorem 3. *Let* $(X,\rho,\preceq)$ *be a partially ordered complete metric space,* $(X\times X,d,\preceq)$ *and* $F: X\times X\to X$ *be a continuous map with the mixed monotone property. Let there exists* $\alpha\in[0,1/2)$*, so that the inequality*

$$\rho(F(x,y),F(u,v))\le\alpha\rho(x,F(x,y))+\alpha\rho(u,F(u,v))$$

holds for all $x\succeq u$ *and* $y\preceq v$*. If there exists at least one ordered pair* (x,y)*, such that* $x\preceq F(x,y)$ *and* $y\succeq F(y,x)$*, then there exists a coupled fixed point* (x,y) *of* F*.*

If in addition every pair of elements in $X\times X$ *has an lower or an upper bound, then the coupled fixed point is unique.*

Example 1. *Let* $X=\ell_1$*, endowed with its classical norm* $\|x\|_1=\sum_{i=1}^{\infty}|x_i|$ *and the metric* $\rho_1(x,y)=\|x-y\|$*. Let us define a partial order in* X *by* $x\preceq y$*, if* $|x_i|\le|y_i|$ *for all* $i\in\mathbb{N}$*. Let us define* $F: X\times X\to X$ *by* $F(x,y)=\left\{\frac{|x_i|}{2}-\frac{|y_i|}{3}+\frac{1}{2^i}\right\}_{i=1}^{\infty}$*.*

The map F satisfies the conditions of Theorem 3 and consequently F has a coupled fixed point.

It is easy to observe that for any two elements $x,y\in(X,\rho_1,\preceq)$ there exists an element z, which is comparable with both of them (we can choose $z_i\ge\max\{|x_i|,|y_i|\}$). Thus the coupled fixed point is unique.

Coupled fixed points results for Chatterjea type of maps with the mixed monotone property obtained with the help of a variational technique

Theorem 4. *Let* $(X,\rho,\preceq)$ *be a partially ordered complete metric space,* $(X\times X,d,\preceq)$ *and* $F: X\times X\to X$ *be a continuous map with the mixed monotone property. Let there exists* $\alpha\in[0,1/2)$*, so that the inequality*

$$\rho(F(x,y),F(u,v))\le\alpha\rho(x,F(u,v))+\alpha\rho(u,F(x,y))$$

holds for all $x\succeq u$ *and* $y\preceq v$*. If there exists at least one ordered pair* (x,y)*, such that* $x\preceq F(x,y)$ *and* $y\succeq F(y,x)$*, then there exists a coupled fixed point* (x,y) *of* F*.*

If in addition every pair of elements in $X\times X$ *has an lower or an upper bound, then the coupled fixed point is unique.*

Coupled fixed points results for Hardy–Rogers type of maps with the mixed monotone property obtained with the help of a variational technique

Theorem 5. *Let $(X, \rho, \preceq)$ be a partially ordered complete metric space, $(X \times X, d, \preceq)$ and $F : X \times X \to X$ be a continuous map with the mixed monotone property. Let there exists $\alpha + \beta + \gamma \in [0, 1/2)$, so that the inequality*

$$\begin{aligned}\rho(F(x,y),F(u,v)) \quad &\leq \quad \alpha(\rho(x,u)+\rho(y,v))+\beta(\rho(x,F(x,y)+\rho(u,F(u,v))\\ &\quad +\gamma(\rho(x,F(u,v))+\gamma\rho(u,F(x,y)).\end{aligned}$$

holds for all $x \succeq u$ and $y \preceq v$. If there exists at least one ordered pair (x, y), such that $x \preceq F(x, y)$ and $y \succeq F(y, x)$, then there exists a coupled fixed point (x, y) of F.

If in addition every pair of elements in $X \times X$ has an lower or an upper bound, then the coupled fixed point is unique.

If we take $\beta = \gamma = 0$ in Theorem 5 we get Theorem 2. If we take $\alpha = \gamma = 0$ in Theorem 5 we get Theorem 3. If we take $\alpha = \beta = 0$ in Theorem 5 we get Theorem 4.

Chapter II
ERROR ESTIMATES FOR COUPLED BEST PROXIMITY POINTS

One of the advantage of Banach fixed point Theorem is the error estimates of the successive iterations and the rate of convergence.

One kind of a generalization of the Banach Contraction Principle is the notation of cyclical maps [36], i.e. $T(A) \subseteq B$ and $T(B) \subseteq A$. Because a non-self mapping $T : A \to B$ does not necessarily have a fixed point, one often attempts to find an element x which is in some sense closest to Tx, i.e. we try to solve the problem $\min\{\rho(x, Tx) : x \in A\}$. Best proximity point theorems are relevant in this perspective. The notation of best proximity point is introduced in [22].

In contrast with all the results about fixed points for self maps and cyclic maps, where "a priori error estimates" and "a posteriori error estimates" are obtained there were no such results about best proximity points.

We have obtained "a priori error estimates" and "a posteriori error estimates" in [52] for the cyclic contractions, investigated in [22].

On the technique of obtaining error estimates of best proximity points

The technique of the next results from [52] will be used in the sequel for obtaining of error estimates of the coupled best proximity points for different kind of maps.

Theorem 6. *Let A and B be nonempty, closed and convex subsets of a uniformly convex Banach $(X, \|\cdot\|)$ space, such that $d = \operatorname{dist}(A, B) > 0$, and let there exist $C > 0$ and $q \geq 2$, such that $\delta_{\|\cdot\|}(\varepsilon) \geq C\varepsilon^q$. Let $T : A \cup B \to A \cup B$ be a cyclic contraction map and T satisfies the inequality $\rho(Tx, Ty) \leq k\rho(x, y) + (1 - k)\operatorname{dist}(A, B)$ for any $x \in A$, $y \in B$ and for some $k \in (0, 1)$. Then*

(i) *there exists a unique best proximity point ξ of T in A, $T\xi$ is a unique best proximity point of T in B and $\xi = T^2\xi = T^{2n}\xi$;*

(ii) *for any* $x_0 \in A$ *the sequence* $\{x_{2n}\}_{n=1}^{\infty}$ *converges to* ξ *and* $\{x_{2n+1}\}_{n=1}^{\infty}$ *converges to* $T\xi$, *where* $x_{n+1} = Tx_n$, $n = 0, 1, 2, \dots$;

(iii) *a priori error estimate holds* $\|\xi - T^{2n}x\| \le \frac{\|x-Tx\|}{1-\sqrt[q]{k^2}} \sqrt[q]{\frac{\|x-Tx\|-d}{Cd}} \left(\sqrt[q]{k}\right)^{2n}$

(iv) *a posteriori error estimate holds* $\|T^{2n}x - \xi\| \le \frac{\|T^{2n-1}x-T^{2n}x\|}{1-\sqrt[q]{k^2}} \sqrt[q]{\frac{\|T^{2n-1}x-T^{2n}x\|-d}{Cd}} \sqrt[q]{k}$.

Error estimates for coupled fixed points and coupled best proximity points of cyclic contraction maps

The main result about coupled best proximity points from [27, 49], applied in uniformly convex Banach space is summarized in the next theorem, which one enriches the results from [27, 49].

Theorem 7. *Let* A *and* B *be nonempty closed and convex subsets of a uniformly convex Banach space* $(X, \|\cdot\|)$. *Let* $F : A \times A \to B$, $G : B \times B \to A$ *and the ordered pair* (F, G) *be such that there are non–negative numbers* α, β, *so that* $\alpha + \beta < 1$ *and there holds the inequality*

$$\rho(F(x, y), G(u, v)) \le \alpha\rho(x, u) + \beta\rho(y, v) + (1 - (\alpha + \beta))\mathrm{dist}(A, B)$$

for all $(x, y) \in A \times A$ *and* $(u, v) \in B \times B$. *Then* F *has a unique coupled best proximity point* $(\xi, \eta) \in A \times A$ *and* G *has a unique coupled best proximity point* $(\zeta, \varsigma) \in B \times B$, *(i.e.* $\|\xi - F(\xi, \eta)\| = \|\eta - F(\eta, \xi)\| = d$ *and* $\|\zeta - G(\zeta, \varsigma)\| = \|\varsigma - G(\varsigma, \zeta)\| = d$*). Moreover there hold*

$$G(F(\xi, \eta), F(\eta, \xi)) = \xi, \quad G(F(\eta, \xi), F(\xi, \eta)) = \eta,$$
$$F(G(\zeta, \varsigma), G(\varsigma, \zeta)) = \zeta, \quad F(G(\varsigma, \zeta), F(\zeta, \varsigma)) = \varsigma$$

and

$$\zeta = F(\xi, \eta), \;\; \varsigma = F(\eta, \xi), \;\; \xi = G(\zeta, \varsigma), \;\; \eta = G(\varsigma, \zeta).$$

For any arbitrary point (x_0, y_0) *there hold* $\lim\limits_{n\to\infty} x_{2n} = \xi$, $\lim\limits_{n\to\infty} y_{2n} = \eta$, $\lim\limits_{n\to\infty} x_{2n+1} = \zeta$, $\lim\limits_{n\to\infty} y_{2n+1} = \varsigma$ *and* $\|\xi - \zeta\| + \|\eta - \varsigma\| = 2d$.

Error estimates for coupled best proximity points for cyclic contraction maps

Theorem 8. *Let* A *and* B *be nonempty closed and convex subsets of a uniformly convex Banach space with modulus of convexity of power type with constants* $C > 0$ *and* $q > 1$. *Let* $F : A \times A \to B$, $G : B \times B \to A$ *and the ordered pair* (F, G) *be such that there are non–negative numbers* α, β, *so that* $\alpha + \beta < 1$ *and there holds the inequality*

$$\rho(F(x, y), G(u, v)) \le \alpha\rho(x, u) + \beta\rho(y, v) + (1 - (\alpha + \beta))\mathrm{dist}(A, B) \tag{3}$$

for all $(x, y) \in A \times A$ *and* $(u, v) \in B \times B$. *Then*

(i) F *has a unique coupled best proximity point* $(\xi, \eta) \in A \times A$ *and* G *has a unique coupled best proximity point* $(\zeta, \varsigma) \in B \times B$, *(i.e.* $\|\xi - F(\xi, \eta)\| = \|\eta - F(\eta, \xi)\| = d$ *and* $\|\zeta - G(\zeta, \varsigma)\| = \|\varsigma - G(\varsigma, \zeta)\| = d$*). Moreover* $\zeta = F(\xi, \eta)$, $\varsigma = F(\eta, \xi)$, $\xi = G(\zeta, \varsigma)$ *and* $\eta = G(\varsigma, \zeta)$. *For any arbitrary point* (x_0, y_0) *there hold* $\lim_{n\to\infty} x_{2n} = \xi$, $\lim_{n\to\infty} y_{2n} = \eta$, $\lim_{n\to\infty} x_{2n+1} = \zeta$, $\lim_{n\to\infty} y_{2n+1} = \varsigma$ *and* $\|\xi - \zeta\| + \|\eta - \varsigma\| = 2d$;

(ii) *a priori error estimates hold*

$$\max\left\{\|\xi - x_{2m}\|, \|\eta - y_{2m}\|\right\} \le P_{0,1}(x,y)\sqrt[q]{\frac{W_{0,1}(x,y)}{Cd}} \cdot \frac{\left(\sqrt[q]{(\alpha+\beta)^2}\right)^m}{1-\sqrt[q]{(\alpha+\beta)^2}}$$

(iii) *a posteriori error estimates hold*

$$\max\left\{\|\xi - x_{2n}\|, \|\eta - y_{2n}\|\right\} \le P_{2n,2n-1}(x,y)\sqrt[q]{\frac{W_{2n,2n-1}(x,y)}{Cd}} \cdot \frac{\sqrt[q]{\alpha+\beta}}{1-\sqrt[q]{(\alpha+\beta)^2}},$$

where $\{x_n\}_{n=0}^{\infty}$ *and* $\{y_n\}_{n=0}^{\infty}$ *are the sequences defined in Definition 11.*

Error estimates for coupled fixed points for cyclic contraction maps

It is proved in [27, 49] that if $d = 0$ then there exists a unique common coupled fixed point for F and G. We extend the results from [27, 49] by dropping the assumption $d = 0$.

Theorem 9. *Let A and B be nonempty closed subsets of a complete metric space (X, ρ) and $F : A \times A \to B$ and $G : B \times B \to A$. Let there exist $\alpha, \beta > 0$, $\alpha + \beta < 1$, such that*

$$\rho(F(x,y), G(u,v)) \le \alpha\rho(x,u) + \beta\rho(y,v)$$

for all $x, y \in A$ and $u, v \in B$. Then

(i) *there exists a unique pair (ξ, η) in $A \cap B$, which is a common coupled fixed point for the maps F and G and moreover the iteration sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$, defined in Definition 11 converge to ξ and η respectively*

(ii) *a priori error estimates hold* $\max\left\{\rho(x_n,\xi), \rho(y_n,\eta)\right\} \le \frac{(\alpha+\beta)^n}{1-\alpha-\beta}(\rho(x_1,x_0) + \rho(y_1,y_0))$

(iii) *a posteriori error estimates hold*

$$\max\left\{\rho(x_n,\xi), \rho(y_n,\eta)\right\} \le \frac{\alpha+\beta}{1-(\alpha+\beta)}(\rho(x_{n-1},x_n) + \rho(y_{n-1},y_n))$$

(iv) *the rate of convergence for the sequences of successive iterations is given by*

$$\rho(x_n,\xi) + \rho(y_n,\eta) \le (\alpha+\beta)\left(\rho(x_{n-1},\xi) + (y_{n-1},\eta)\right).$$

If the functions F and G depend only on its first variable and $\beta = 0$ we get that the pair (F, G) is a cyclic contraction in the sense of [22, 36], i.e. $Tx = F(x,y) : A \to B$ and $Tu = G(u,v) : B \to A$. The results from [22, 52] are corollaries of Theorem 8 and the ones from [36, 46] are corollaries from Theorem 9.

Application of Theorem 8 in solving of systems of integral equations

We will illustrate Theorem 8 with some examples.

Example 2. *Let us consider the space $L_2[0,1]$ of all measurable functions with an integrable square, endowed with the canonical norm $\|f\|_2 = \sqrt{\int_0^1 f^2(t)dt}$. Let $A = \{f \in L_2[0,1] : f(t) \geq t \text{ for all } t \in [0,1]\}$. Let us search for the solutions of the system*

$$(4) \qquad \left| \begin{array}{rcl} \displaystyle\int_0^1 \left(x(t) + \frac{3t}{2}\int_0^1 s\frac{x(s)+y(s)}{2}ds + \frac{t}{2}\right)^2 dt & = & \frac{4}{3} \\ \displaystyle\int_0^1 \left(y(t) + \frac{3t}{2}\int_0^1 s\frac{x(s)+y(s)}{2}ds + \frac{t}{2}\right)^2 dt & = & \frac{4}{3}, \end{array} \right.$$

which belong to the set A.

Let us denote $B = \{f \in L_2[0,1] : f(t) \leq -t \text{ for all } t \in [0,1]\}$. It is easy to observe that $d = \rho(A,B) = \frac{2}{\sqrt{3}}$. Let $F(x,y) = -\left(\frac{3t}{2}\int_0^1 s\frac{x(s)+y(s)}{2}ds + \frac{t}{2}\right)$ for any $(x,y) \in A \times A$ and $G(u,v) = \frac{3t}{2}\int_0^1 s\frac{|u(s)|+|v(s)|}{2}ds + \frac{t}{2}$ for any $(u,v) \in B \times B$. The pair $(\xi,\eta) \in A \times A$ is a solution of (4) if and only if (ξ,η) is a coupled best proximity points (ξ,η) of F in $A \times A$.

Table 1. Number $2m$ of iterations needed by the a posteriori estimate

ε	0.1	0.01	0.001	0.0001	0.00001	0.000001
$2m$	15	21	28	34	41	47

Application of Theorem 8 in solving of particular systems of linear equations

The examples that we will consider are from [27, 49] and are in the space $(\mathbb{R}, |\cdot|)$.

Let us point out that the modulus of convexity $\delta_{\|\cdot\|}$ is considered if the Banach space is at least two dimensional. As far as $\mathbb{R}$, endowed with its canonical norm is a subspace of $\mathbb{R}_2^2$ we get that $\delta_{(\mathbb{R},|\cdot|)}(\varepsilon) \geq \delta_{(\mathbb{R}_2^2,\|\cdot\|_2)}(\varepsilon) = \frac{\varepsilon^2}{8}$. It is easy to observe that in $\mathbb{R}$ there holds the equality $\delta_{(\mathbb{R},|\cdot|)}(\varepsilon) = \frac{\varepsilon}{2}$. Indeed $B_{(\mathbb{R},|\cdot|)} = [-1,1]$. Then

$$\delta_{(\mathbb{R},|\cdot|)}(\varepsilon) = \inf\left\{\left|1 - \frac{x+y}{2}\right| : x,y \in [-1,1], |x-y| \geq \varepsilon\right\}.$$

The infimum is attained, when $x = 1$ and $y = 1-\varepsilon$. Therefore $\delta_{(\mathbb{R},|\cdot|)}(\varepsilon) = \left|1 - \frac{1+(1-\varepsilon)}{2}\right| = \frac{\varepsilon}{2}$.

Example 3. *([49]) Let us consider the space $\mathbb{R}$, endowed with the canonical norm $|\cdot|$ and $A = [1,2]$. We search for the solutions of the system*

$$\left| \begin{array}{rcl} 5x + y & = & 6 \\ 5y + x & = & 6, \end{array} \right.$$

which belong to the set A.

Let us denote $B = [-2, -1]$. Let $F : A \times A \to B$ and $G : B \times B \to A$ be defined by $F(x, y) = \frac{-x-y-2}{4}$ and $G(x, y) = \frac{-x-y+2}{4}$. The ordered pair (F, G) satisfies the conditions of Theorem 8 with $\alpha = \beta = 1/4$ and $d = d(A, B) = 2$ [49]. The couple $(1, 1)$ is a unique couple of best proximity points of F in A.

Table 2. Number $2m$ of iterations needed by the a priori estimate

ε	0.1	0.01	0.001	0.0001	0.00001	0.000001
$2m$	8	12	16	18	22	26

Table 3. Number $2m$ of iterations needed by the a posteriori estimat

ε	0.1	0.01	0.001	0.0001	0.00001	0.000001
$2m$	16	36	66	120	172	232

Modified coupled fixed points and coupled best proximity points

Deep results in the theory of coupled fixed points can be found for example in [7, 8, 9, 10]. We have tried to enrich the known results about coupled best proximity points for order pairs of cyclic contraction maps (F, G), by proving that the coupled best proximity points $(x, y) \in A \times A$ reduce to the point $(x, x) \in A \times A$. This result shows why the application made in [28] is valid only for symmetric linear systems and cannot be extended to an application for solving arbitrary linear systems.

In order to get a general result for the existence of coupled best proximity points $(x, y) \in A \times A$ with $x \neq y$, we needed to consider an ordered pair of an order pair of maps $((F, f), (G, g))$, such that $F : A_1 \times A_2 \to B_1$, $f : A_1 \times A_2 \to B_2$, $G : B_1 \times B_2 \to A_1$, $g : B_1 \times B_2 \to A_2$, where $A_1, A_2, B_1, B_2 \subset X$.

Just to fit some of the formulas in the text field let us denote $d_x = \mathrm{disr}(A_x, B_x)$ and $d_y = \mathrm{dist}(A_y, B_y)$.

Definition 25. *Let A_x, A_y, B_x and B_y be nonempty subsets of a metric space (X, ρ), $F : A_x \times A_y \to B_x$, $f : A_x \times A_y \to B_y$, $G : B_x \times B_y \to A_x$ and $g : B_x \times B_y \to A_y$. The ordered pair of orderer pairs $((F, f), (G, g))$ is said to be a cyclic contraction ordered pair if there exist non-negative numbers $\alpha, \beta, \gamma, \delta$, satisfying* $\max\{\alpha + \gamma, \beta + \delta\} < 1$ *and the inequality*

$$\begin{aligned} S_1 &= \rho(F(x, y), G(u, v)) + \rho(f(z, w), g(t, s)) \\ &\leq \alpha\rho(x, u) + \beta\rho(y, v) + \gamma\rho(z, t) + \delta\rho(w, s) + (1 - (\alpha + \gamma))d_x + (1 - (\beta + \delta))d_y \end{aligned}$$

for all $(x, y), (z, w) \in A_x \times A_y$ and $(u, v), (t, s) \in B_x \times B_y$.

Definition 26. *Let A_x, A_y, B_x and B_y be nonempty subsets of a metric space (X, ρ), $F : A_x \times A_y \to B_x$, $f : A_x \times A_y \to B_y$. An ordered pair $(\xi, \eta) \in A_x \times A_y$ is called a coupled best proximity point of (F, f) in $A_x \times A_y$ if $\rho(\xi, F(\xi, \eta)) = \mathrm{dist}(A_x, B_x)$ and $\rho(\eta, f(\xi, \eta)) = \mathrm{dist}(A_y, B_y)$.*

Definition 27. *Let A_x, A_y, B_x and B_y be nonempty subsets of X. Let $F : A_x \times A_y \to B_x$, $f : A_x \times A_y \to B_y$, $G : B_x \times B_y \to A_x$ and $g : B_x \times B_y \to A_y$. For any pair $(x, y) \in A_x \times A_y$ we define the sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ by $x_0 = x$, $y_0 = y$ and*

$$\begin{array}{ll} x_{2n+1} = F(x_{2n}, y_{2n}), & y_{2n+1} = f(x_{2n}, y_{2n}) \\ x_{2n+2} = G(x_{2n+1}, y_{2n+1}), & y_{2n+2} = g(x_{2n+1}, y_{2n+1}) \end{array}$$

for all $n \geq 0$.

Everywhere in this subsection, when considering the sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ we will assume that they are the sequences defined in Definition 27.

If we put $A_x = A_y = A$, $B_x = B_y = B$, $D = d(A, B)$, $f(x, y) = F(y, x)$, $g(x, y) = G(y, x)$, $z = y$, $w = x$, $t = v$, $s = u$, $\gamma = \beta$ and $\delta = \alpha$ in Definition 25, then we get the maps, investigated in [27, 49].

Comments on the known results about coupled best proximity points

It is interesting to see that in the examples from [27, 49] there holds $\xi = \eta$. It turns out that this is not just a coincidence.

Theorem 10. *Let there hold the assumptions of Theorem 7. Then F has a unique coupled best proximity point $(\xi, \eta) \in A \times A$ and $\xi = \eta$ and G has a unique coupled best proximity point $(\zeta, \varsigma) \in B \times B$.*

The next result enriches the results from [29] by proving that the coupled fixed point (ξ, η) in $A \cap B$ satisfies $\xi = \eta$.

Theorem 11. *Let A and B be nonempty closed subsets of a complete metric space (X, ρ) and $F : A \times A \to B$ and $G : B \times B \to A$. Let there exist $\alpha, \beta > 0$, $\alpha + \beta < 1$, such that $\rho(F(x, y), G(u, v)) \leq \alpha\rho(x, u) + \beta\rho(y, v)$ for all $x, y \in A$ and $u, v \in B$. Then there exists a unique pair (ξ, η) in $A \cap B$, which is a common coupled fixed point for the maps F and G. Moreover the iteration sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ for any arbitrary initial guess $(x, y) \in A \times A$, defined in Definition 11 converge to ξ and η respectively and moreover $\xi = \eta$.*

Modified coupled fixed points

Theorem 12. *Let A_x, A_y, B_x and B_y be nonempty subsets of a complete metric space (X, ρ), $F : A_x \times A_y \to B_x$, $f : A_x \times A_y \to B_y$, $G : B_x \times B_y \to A_x$ and $g : B_x \times B_y \to A_y$. Let there exist $\alpha, \beta, \gamma, \delta > 0$, $\max\{\alpha + \gamma, \beta + \delta\} < 1$, such that*

$$\rho(F(x, y), G(u, v)) + \rho(f(z, w), g(t, s)) \leq \alpha\rho(x, u) + \beta\rho(y, v) + \gamma\rho(z, t) + \delta\rho(w, s)$$

for all $(x, y) \in A_x \times A_y$, $(u, v) \in B_x \times B_y$, $(z, w) \in A_x \times A_y$ and $(t, s) \in B_x \times B_y$. Then

(i) *there exists a unique pair (ξ, η) in $A \cap B$, which is a common coupled fixed point for the maps F and G and the iteration sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$, defined in Definition 27 converge to ξ and η respectively.*

(ii) *a priori error estimates hold* $\max\{\rho(x_n, \xi), \rho(y_n, \eta)\} \leq \frac{k^n}{1-k}(\rho(x_1, x_0) + \rho(y_1, y_0))$

(iii) *a posteriori error estimates hold* $\max\left\{\rho(x_n,\xi),\rho(y_n,\eta)\right\} \leq \frac{k}{1-k}(\rho(x_{n-1},x_n)+\rho(y_{n-1},y_n))$

(iv) *the rate of convergence for the sequences of successive iterations is given by*

$$\rho(x_n,\xi)+\rho(y_n,\eta) \leq k\left(\rho(x_{n-1},\xi)+(y_{n-1},\eta)\right).$$

Modified coupled best proximity points

Just to fit some of the formulas in the text field we will use the notations. Let A_x, A_y, B_x and B_y be nonempty subsets of a metric space (X,ρ). Let us denote $d_x = \operatorname{dist}(A_x,B_x)$, $d_y = \operatorname{dist}(A_y,B_y)$, $d = d_x + d_y$, $P_{n,m}(x,y) = \|x_n - x_m\| + \|y_n - y_m\|$ and $W_{n,m}(x,y) = P_{n,m}(x,y) - (d_x + d_y) = \|x_n - x_m\| + \|y_n - y_m\| - (d_x + d_y)$, where $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be the sequences defined in Definition 27 and $k = \max\{\alpha+\gamma,\beta+\delta\}$, where $\alpha,\beta,\gamma,\delta$ are the constants from Definition 25.

Theorem 13. *Let A_x, A_y, B_x and B_y be nonempty convex subsets of a uniformly convex Banach space $(X,\|\cdot\|)$, $F : A_x \times A_y \to B_x$, $f : A_x \times A_y \to B_y$, $G : B_x \times B_y \to A_x$ and $g : B_x \times B_y \to A_y$. Let the ordered pair $((F,f),(G,g))$ be a cyclic contraction. Then (F,f) has a unique coupled best proximity point $(\xi,\eta) \in A_x \times A_y$ and (G,g) has a unique coupled best proximity point $(\zeta,\varsigma) \in B_x \times B_y$, (i.e. $\|\xi - F(\xi,\eta)\| = d_x$, $\|\eta - f(\xi,\eta)\| = d_y$ and $\|\zeta - G(\zeta,\varsigma)\| = d_x$, $\|\varsigma - g(\zeta,\varsigma)\| = d_y$). Moreover $\zeta = F(\xi,\eta)$, $\varsigma = f(\xi,\eta)$, $\xi = G(\zeta,\varsigma)$ and $\eta = g(\zeta,\varsigma)$. For any arbitrary point $(x,y) \in A \times A$ there hold $\lim_{n\to\infty} x_{2n} = \xi$, $\lim_{n\to\infty} y_{2n} = \eta$, $\lim_{n\to\infty} x_{2n+1} = \zeta$, $\lim_{n\to\infty} y_{2n+1} = \varsigma$ and $\|\xi - \zeta\| + \|\eta - \varsigma\| = d_x + d_y$. Moreover there hold*

$$G(F(\xi,\eta),f(\xi,\eta)) = \xi,\ g(F(\xi,\eta),f(\xi,\eta)) = \eta,$$
$$F(G(\zeta,\varsigma),g(\zeta,\varsigma)) = \zeta,\ f(G(\zeta,\varsigma),g(\zeta,\varsigma)) = \varsigma.$$

If in addition $(X,\|\cdot\|)$ has a modulus of convexity of power type with constants $C > 0$ and $q > 1$, then

(i) *a priori error estimates hold*

$$\max\left\{\|\xi - x_{2m}\|,\|\eta - y_{2m}\|\right\} \leq P_{0,1}(x,y)\sqrt[q]{\frac{W_{0,1}(x,y)}{Cd_x}} \cdot \frac{\sqrt[q]{k^{2m}}}{1-\sqrt[q]{k^2}}$$

(ii) *a posteriori error estimates hold*

$$\max\left\{\|\xi - x_{2n}\|,\|\eta - y_{2n}\|\right\} \leq P_{2n,2n-1}(x,y)\sqrt[q]{\frac{W_{2n,2n-1}(x,y)}{Cd_x}} \cdot \frac{\sqrt[q]{k}}{1-\sqrt[q]{k^2}}$$

Applications of Theorem 13 for solving of systems of transcendent equations

If put $A_x = A_y = A$, $B_x = B_y = B$, $f(x,y) = F(y,x)$, $g(x,y) = G(y,x)$, $z = y$, $w = x$, $t = v$, $s = u$, $\gamma = \beta$ and $\delta = \alpha$, then we get the results from ([29], Theorem 2 and Theorem 3) as corollaries of Theorem 13 and Theorem 12.

We will illustrate Theorem 13 by solving the next system.

Example 4. *Let us consider the system of nonlinear equations:*

$$(5)\qquad \left| \begin{array}{rcrcl} 36x & + & e^y & = & e+68 \\ 4\arctan\left(\frac{x}{2}\right) & + & 18y & = & \pi+18. \end{array} \right.$$

Let us consider the functions

$$F(x,y)=-\frac{x}{8}-\frac{e^y}{32}+\frac{e-60}{32},\quad G(x,y)=-\frac{x}{8}-\frac{e^y}{32}-\frac{e-60}{32},$$

$$f(x,y)=-\frac{\arctan\left(\frac{x}{2}\right)}{4}-\frac{y}{8}+\frac{\pi-14}{16},\quad g(x,y)=-\frac{\arctan\left(\frac{x}{2}\right)}{4}-\frac{y}{8}-\frac{\pi-14}{16}.$$

It is easy to check that $F:[2,+\infty)\times[1,1.5]\to(-\infty,-2]$, $f:[2,+\infty)\times[1,1.5]\to[-1.5,-1]$, $G:(-\infty,-2]\times[-1.5,-1]\to[2,+\infty)$, $g:(-\infty,-2]\times 5-1.5,-1]\to[1,1.5]$ and problem of finding the best proximity points of (F,f) is equivalent to (5).

The ordered pair $((F,f),(G,g))$ is a cyclic contraction with constants $\frac{1}{8}$, $\frac{e^{1.5}}{32}$, $\frac{1}{16}$, $\frac{1}{8}$ and the unique solution of (5) is $(2,1)$.

Table 4. Number $2m$ of iterations needed by the a priori estimate

ε	0.1	0.01	0.001	0.0001	0.00001	0.000001
$2m$	4	6	8	10	12	14

Table 5. Number $2m$ of iterations needed by the a posteriori estimate

ε	0.1	0.01	0.001	0.0001	0.00001	0.000001
$2m$	4	8	12	14	16	20

If we try to solve system (5) with the help of Maple 18.00, we get as an answer

$$x=2\tan(RootOf(72\tan(\_Z)+e^{-\frac{2}{9}\_Z+\frac{1}{18}\pi+1}-e-72))$$

$$y=-\frac{2}{9}RootOf(72\tan(\_Z)+e^{-\frac{2}{9}\_Z+\frac{1}{18}\pi+1}-e-72)+\frac{\pi}{18}+1.$$

If we try numerically to approximate the solutions of the system (5), with the help of Maple 18.00, we get as an answer $\{x=2.000000000, y=.9999999998\}$.

Application of Theorem 13 in solving of systems of linear equations

We will present an application of the results for modified coupled best proximity points in solving systems of linear equations, generalizing the results from [28].

Example 5. *Let $p,q,r,m,n,k>0$ and let us consider the system of linear equations:*

$$(6)\qquad \left| \begin{array}{rcl} px+qy & = & r \\ my+nx & = & k. \end{array} \right.$$

Let us assume that $p\geq q$ and $m\geq n$. Let $a=\frac{kq-nr}{mq-np}$, $b=-\frac{rp-mr}{mq-np}$ be the solutions of the system (6).

Let $\mu > 0$ be such that $\max\left\{\mu + \frac{(1+\mu)q}{p}, \mu + \frac{(1+\mu)n}{m}\right\} < 1$.

Let us consider the functions

$$F(x,y) = -\mu x - \frac{(1+\mu)q}{p}y + \frac{(1+\mu)r}{p} - 2a, \quad G(x,y) = -\mu x - \frac{(1+\mu)q}{p}y - \frac{(1+\mu)r}{p} + 2a,$$

$$f(x,y) = -\mu y - \frac{(1+\mu)n}{m}x + \frac{(1+\mu)k}{m} - 2b, \quad g(x,y) = -\mu y - \frac{(1+\mu)n}{m}x - \frac{(1+\mu)k}{m} + 2b.$$

The ordered pair $((F,f),(G,g))$ is a cyclic contraction with constants μ, $\frac{(1+m)(q)}{p}$, μ, $\frac{(1+m)(n)}{m}$. The rate of convergence depends on the constant $\max\left\{\mu + \frac{(1+\mu)q}{p}, \mu + \frac{(1+\mu)n}{m}\right\}$, which is an increasing function of μ. That is why by choosing a smaller $\mu \in (0,1)$ we will get faster convergence. It is not possible to choose $\mu = 0$ and therefore the upper bound of the convergence $\max\left\{\frac{q}{p}, \frac{n}{m}\right\}$ could not be obtained.

We will consider a particular case of Example 5.

Example 6. *Let us consider the space $\mathbb{R}$, endowed with the canonical norm $|\cdot|$. We search for the solutions of the system*

$$\left|\begin{array}{rcl} 2x + y & = & 12 \\ 3y + x & = & 11. \end{array}\right. \tag{7}$$

Table 6. Number $2m$ of iterations needed by the a priori estimate

ε	0.1	0.01	0.001	0.0001	0.00001	0.000001
$\mu = 1/4$	48	66	82	100	116	134
$\mu = 1/8$	48	64	82	98	116	134
$\mu = 1/80$	46	64	80	98	116	132

Table 7. Number $2m$ of iterations needed by the a posteriori estimate

ε	0.1	0.01	0.001	0.0001	0.00001	0.000001
$\mu = 1/4$	210	406	666	1036	1432	1982
$\mu = 1/8$	56	106	190	276	378	496
$\mu = 1/80$	22	45	78	106	154	210

Existence and uniqueness of coupled fixed points and coupled best proximity points of p–cyclic contractions

Let A_i, $i = 1, 2, \ldots, p$ be nonempty sets. Just to simplify some of the notations we will assume that $A_{p+k} = A_k$ for $k = 1, 2, \ldots, p$.

The notion of a coupled best proximity point for cyclic maps was defined in [49] and the notion of coupled best proximity point for p–cyclic maps was introduced in [33]. We will combine both definitions to define a coupled best proximity point for a p–cyclic maps.

Definition 28. *Let A_i, $i = 1, 2, \ldots, p$ be nonempty subsets of a metric space (X, ρ) we call a map $T \cup_{i=1}^{p} (A_i \times A_i) \to \cup_{i=1}^{p}(A_i \times A_i)$ a p–cyclic map if $T : A_i \times A_i \to A_{i+1}$ for $i = 1, 2, \ldots, p$.*

Definition 29. *Let A_i, $i = 1, 2, \dots, p$ be nonempty subsets of a metric space (X, ρ) and T be a p–cyclic map. A point $(x, y) \in A_i \times A_i$ is said to be a best proximity point of T in $A_i \times A_i$, if $\rho(x, T(x, y)) = \rho(y, T(y, x)) = \mathrm{dist}(A_i, A_{i+1})$.*

Following [26] we will define an iterated sequence $\{(x_n, y_n)\}_{n=0}^{\infty}$, generated by a p–cyclic map T.

Definition 30. *Let $\{A_i\}_{i=1}^{p}$ be nonempty subsets of a metric space (X, ρ) and T be a p–cyclic map. Let for every $(x_0, y_0) \in A_i \times A_i$ we define the sequence (x_n, y_n) inductively by $(x_1, y_1) = (T(x_0, y_0), T(y_0, x_0))$ and if (x_n, y_n) has been already defined then $(x_{n+1}, y_{n+1}) = (T(x_n, y_n), T(y_n, x_n))$.*

Following [33] we will define a cyclic contractive condition for a p–cyclic map $T : A_i \times A_i \to A_{i+1}$ for $i = 1, 2, \dots, p$.

Definition 31. *Let $\{A_i\}_{i=1}^{p}$ be nonempty subsets of a metric space (X, ρ) and T be a p–cyclic map. The map T is called p–cyclic contraction. If there exist $\alpha, \beta \geq 0$, $\alpha + \beta \in (0, 1)$, such that the inequality*

$$\rho(T(x, y), T(u, v)) \leq \alpha\rho(x, u) + \beta\rho(y, v) + (1 - (\alpha + \beta))\mathrm{dist}(A_i, A_{i+1})$$

holds for every $(x, y) \in A_i \times A_i$, $(u, v) \in A_{i+1} \times A_{i+1}$, $1 \leq i \leq p$.

The next lemma is a generalization of the results from [33], where authors have proven that in the case of a one variable maps T, that the distances between the successive sets should be equal.

Lemma 3. *Let $\{A_i\}_{i=1}^{p}$ be nonempty subsets of a metric space X and T be a p–cyclic contraction map. Then $\mathrm{dist}(A_i, A_{i+1}) = \mathrm{dist}(A_{i+1}, A_{i+2})$ for $i = 1, 2, \dots, p$.*

By Lemma 3 it follows that whenever we consider a p–cyclic map T, then the distances between the consecutive sets are equal and we can use the notation $d = d(A_i, A_{i+1}) = d(A_{i-1}, A_i) = \ldots = d(A_1, A_2)$.

Coupled fixed points for p–cyclic maps

Theorem 14. *Let $A_1, A_2, \dots, A_p$ be nonempty, closed and convex subsets of a complete metric space (X, ρ). Let T be a p–cyclic map. If here exist $\alpha, \beta \geq 0$, $\alpha + \beta \in (0, 1)$, such that the inequality $d(T(x, y), T(u, v)) \leq \alpha d(x, u) + \beta d(y, v)$ holds for every $(x, y) \in A_i \times A_i$, $(u, v) \in A_{i+1} \times A_{i+1}$, $1 \leq i \leq p$. Then there exists a unique order pair $(z, v) \in \cap_{i=1}^{p}(A_i \times A_i)$, such that, if $(x_0, y_0) \in A_i \times A_i$ be an arbitrary point of $A_i \times A_i$, the sequence $\{(x_n, y_n)\}\}_{n=0}^{\infty}$ converges to (z, v) and the order pair (z, v) is a coupled fixed point of T. Moreover, there hold*

(i) the a priori estimate $\max\{\rho(x_n, z), \rho(y_n, z)\} \leq \dfrac{\gamma^n}{1-\gamma}(\rho(x_1, x_0) + \rho(y_1, y_0))$

(ii) the a posteriori estimate $\max\{\rho(x_n, z), \rho(y_n, v)\} \leq \dfrac{\gamma}{1-\gamma}(\rho(x_{n-1}, x_n) + \rho(y_{n-1}, y_n))$

(iii) the rate of convergence $\rho(x_n, z) + \rho(y, v) \leq \gamma(\rho(x_{n-1}, z) + \rho(y_{n-1}, v))$,

where $\gamma = \alpha + \beta$.

Coupled best proximity points for p–cyclic contraction maps

Let us recall some of the notations used in the beginning of this section, just to fit some of the formulas in the text field $P_{n,m}(x,y) = \|x_n - x_m\| + \|y_n - y_m\|$ and $W_{n,m}(x,y) = P_{n,m}(x,y) - 2d = \|x_n - x_m\| + \|y_n - y_m\| - 2d$, where $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be the sequences defined in Definition 30.

Theorem 15. *Let $A_1, A_2, \ldots, A_p$ be nonempty, closed and convex subsets of a uniformly convex Banach space $(X, \|\cdot\|)$ with modulus of convexity of power type q with ta constant C. Let T be a p–cyclic contraction. Then there exists a unique order pair $(z_i, v_i) \in A_i \times A_i$ $(1 \le i \le p)$, such that, if $(x,y) \in A_i \times A_i$ is any coupled point of $A_i \times A_i$, the sequence $T^{pn}(x,y)$ converges to (z_i, v_i) and order pair (z_i, v_i) is a best proximity point of T in $A_i \times A_i$. Moreover, if $T^j(z_i, v_i) = z_{i+j}$ and $T^j(v_i, z_i) = v_{i+j}$, then a order pair (z_{i+j}, v_{i+j}) is a coupled best proximity point in $A_{i+j} \times A_{i+j}$ for $j = 1, \ldots, (p-1)$ and (z_i, v_i) is the unique periodic point of T with period p. There hold*
the a priori error estimate

$$\max\{\|\xi - x_{pm}\|, \|\eta - x_{pm}\|\} \le P_{0,1}(x,y)\sqrt[q]{\frac{W_{0,1}(x,y)}{Cd}} \cdot \frac{\left(\sqrt[q]{\gamma}\right)^{pm}}{1 - \sqrt[q]{\gamma^p}}$$

and the a posteriori error estimate

$$\max\{\|\xi - x_{pn}\|, \|\eta - x_{pn}\|\} \le P_{pn,pn-1}(x,y)\sqrt[q]{\frac{W_{pn,pn-1}(x,y)}{Cd}} \frac{\sqrt[q]{\gamma}}{1 - \sqrt[q]{\gamma^p}},$$

where (ξ, η) is the best proximity point of T in A_i for $(x_0, y_0) \in A_i \times A_i$ and $\gamma = \alpha + \beta$.

If $p = 2$, we get as a particular case the results from [29].

Applications of Theorem 15

Let $\varphi, \psi : [1, +\infty) \to [1, +\infty)$ be such that $\max\{\varphi(x), \psi(x)\} \le x$ for any $x \in [1, +\infty)$. Let us define the function $f(x,y) = \lambda + (1-\lambda)(\mu\varphi(x) + (1-\mu)\psi(y))$. Let us consider the system of equations

$$(8) \qquad \left| \begin{array}{rcl} |x|^p + |\lambda + (1-\lambda)(\mu\varphi(x) + (1-\mu)\psi(y))|^p &=& 2 \\ |y|^p + |\lambda + (1-\lambda)(\mu\psi(y) + (1-\mu)\varphi(x))|^p &=& 2 \\ x - f(f(f(x,y), f(y,x)), f(f(y,x), f(x,y))) &=& 0 \\ y - f(f(y,x), f(x,y)), f(f(x,y), f(y,x))) &=& 0 \end{array} \right.$$

for $x, y \ge 0$ and $\lambda, \mu \in (0,1)$.

Let $A_1 = \{(x,0,0) : x \ge 1\}$, $A_2 = \{(0,x,0) : x \ge 1\}$, $A_3 = \{(0,0,x) : x \ge 1\}$ be subsets of $(\mathbb{R}^3, \|\cdot\|_p)$, $p \in (1,\infty)$. Let us define the maps by $T((x,0,0),(y,0,0)) = (0, f(x,y), 0)$; $T((0,x,0),(0,y,0)) = (0,0,f(x,y))$; $T((0,0,x),(0,0,y)) = (f(x,y),0,0)$ for some $\lambda, \mu \in (0,1)$. The map T satisfies the conditions of Theorem 15. Therefore there exist (z,z), which is a coupled best proximity point of T in $A_1 \times A_1$ and it is easy to see that

$z = (1, 0, 0)$. Consequently (z, z) is the unique solution of the system of equations

$$\left| \begin{array}{rcl} \|x - T(x,y)\|_p^p & = & 2 \\ \|y - T(y,x)\|_p^p & = & 2 \\ x - T^3(x,y) & = & 0 \\ y - T^3(y,x) & = & 0, \end{array} \right.$$

which is the solution of (8).

If we try to solve (8) with the use of some Algebraic Computer System, for example Maple 18.00, the software could not find the exact solution even for not too much complicated functions ($p = 2$, $\varphi(x) = x^{1/2}$, $\psi(x) = x$).

If we try to solve it numerically, Maple 18.00 finds that $x = y = 1$, but could not find that this is a solution for every $\lambda, \mu \in (0, 1)$ and presents two approximations of λ and μ.

If we consider the particular case $p = 3$, $\varphi(x) = \sqrt{x}$ and $\psi(x) = \sqrt{\log(x) + 1}$, then Maple 18.00 could not solve (8) even numerically.

Chapter III
COUPLED BEST PROXIMITY POINTS IN MODULAR FUNCTION SPACES

We have tried to generalize the idea of best proximity points in modular function spaces and to present an application for integral operators in Orlicz function spaces, endowed with an Orlicz function modular.

A generalization of Eldred and Veermani's key lemmas in an investigation of best proximity points in modular function spaces

The next lemmas are a generalization of the key lemmas from [22] of Eldred and Veermani in modular function spaces.

Lemma 4. *Let $\rho \in \mathfrak{R}$. Let ρ be (UC1), has the Δ_2–property, $A \subset L_\rho$ be a ρ–closed and convex subset and $B \subset L_\rho$ be ρ–closed subset. If the sequences $\{x_n\}_{n=1}^\infty, \{z_n\}_{n=1}^\infty \subset A$ and $\{y_n\}_{n=1}^\infty \subset B$ be such that:*

(i) $\lim_{n\to\infty} \rho(z_n - y_n) = d_\rho$

(ii) for every $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that for every $m > n \geq N_0$ there holds the inequality $\rho(x_m - y_n) \leq d_\rho + \varepsilon$.

Ten for every $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that for every $m > n \geq N_1$ there holds the inequality $\rho(x_m - z_n) < \varepsilon$.

Corollary 1. *Let $\rho \in \mathfrak{R}$. Let ρ be (UC1), has the Δ_2–property, A be a ρ–closed and convex subset of L_ρ and B be ρ–closed subset of L_ρ. If the sequences $\{x_n\}_{n=1}^\infty, \{z_n\}_{n=1}^\infty \subset A$ and $\{y_n\}_{n=1}^\infty \subset B$ be such that:*

(i) $\lim_{n\to\infty} \rho(z_n - y_n) = d_\rho$

(ii) $\lim_{n\to\infty} \rho(x_n - y_n) = d_\rho$.

Then $\lim_{n\to\infty} \rho(x_n - z_n) = 0$.

Lemma 5. *Let $\rho \in \mathfrak{R}$. Let ρ has the Δ_2–property, be uniformly continuous and $A, B \subset L_\rho$ be subsets. If the sequences $\{y_n\}_{n=1}^\infty \subset B$ and $\{x_n\}_{n=1}^\infty, \{z_n\}_{n=1}^\infty \subset A$ be such that:*

(i) $\lim_{n\to\infty} \rho(z_n - x_n) = 0$

(ii) for every $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that for every $m \geq n \geq N_0$ there holds the inequality $\rho(z_m - y_n) \leq d_\rho + \varepsilon$.

Then for every $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that for every $m \geq n \geq N_1$ there holds the inequality $\rho(x_m - y_n) < d_\rho + \varepsilon$.

We would like to mention that if ρ satisfies the triangle inequality, the proof is trivial and we do not need the assumption that ρ is uniform continuous.

Corollary 2. *Let $\rho \in \mathfrak{R}$. Let ρ has the Δ_2–property, be uniformly continuous and $A, B \subset L_\rho$ be subsets. If the sequences $\{x_n\}_{n=1}^\infty, \{z_n\}_{n=1}^\infty \subset A$ and $\{y_n\}_{n=1}^\infty \subset B$ be such that:*

(i) $\lim_{n\to\infty} \rho(z_n - x_n) = 0$

(ii) $\lim_{n\to\infty} \rho(z_n - y_n) = d_\rho$.

Then $\lim_{n\to\infty} \rho(x_n - y_n) = d_\rho$.

Best proximity points for cyclic ρ contraction maps in modular function spaces

We will generalize the notion of best proximity point in a metric spaces [22] for modular function spaces.

Definition 32. *Let $\rho \in \mathfrak{R}$, $A, B \subset L_\rho$ be two subsets we call a modular distance between the sets A and B the number* $\inf\{\rho(x, y) : x \in A, y \in B\}$ *and we will denote it by $d_\rho(A, B)$.*

Definition 33. *Let $\rho \in \mathfrak{R}$, $A, B \subset L_\rho$ be two subsets and $T : A \cup B \to A \cup B$ be a cyclic map. A point $\xi \in A$ is called a ρ–best proximity point of the cyclic map T in A if $\rho(\xi, T\xi) = d_\rho(A, B)$.*

Definition 34. *Let $\rho \in \mathfrak{R}$, $A, B \subseteq L_\rho$ be subsets. The map $T : A \cup B \to A \cup B$ is called a cyclic ρ–contraction if it is cyclic map and there exist $k \in (0, 1)$, such that the inequality*

$$\rho(Tx - Ty) \leq k\rho(x - y) + (1 - k)d_\rho(A, B)$$

hold for every $x \in A$, $y \in B$.

To simplify the notations we will denote $d_\rho(A, B)$ by d_ρ.

Best proximity points for cyclic ρ–contractions in modular function spaces

The next theorem concerns just best proximity points, but not coupled best proximity points. As far as it is the first result on best proximity points in modular function spaces and we will use its technique we think that it will be good for an easier understanding of the readers.

Theorem 16. *Let $\rho \in \mathfrak{R}$. Let ρ be (UC1), has the Δ_2–property and be uniformly continuous. Let $A, B \subseteq L_\rho$ be ρ–closed, convex subsets, $A \cup B$ be ρ–bounded and $T : A \cup B \to A \cup B$ be a cyclic ρ–contraction. Then there there exists a unique $x \in A$ such that x is a ρ–best proximity point of T in A, $T^2x = x$ and for any $x_0 \in A$ the point x is a ρ–limit of the sequence $\{T^{2n}x_0\}_{n=1}^{\infty}$.*

An application of ρ-best proximity points for integral equations in Orlicz function spaces

Theorem 17. *Let M be an Oricz function, which satisfies the Δ_2 condition and is very convex. Let $L_{\widetilde{M}}$ be the modular function space generated by M. Let $\alpha, \beta, \varphi, \psi \in L_{\widetilde{M}}$. Let us denote $A = \{u \in L_{\widetilde{M}} : \alpha(x) \le u(x) \le \varphi(x)\}$ and $B = \{u \in L_{\widetilde{M}} : \psi(x) \le u(x) \le \beta(x)\}$. Let $K : [0,1] \times [0,1] \to \mathbb{R}$ and $f, g : [0,1] \to \mathbb{R}$. Let the map T be defined by*

$$Tu(x) = -\mathrm{sign}(u(x)) \left(g(x) + \int_0^1 K(x,s) f(u(s)) ds \right).$$

Let the following conditions take place

(i) $\beta(x) < 0 < \alpha(x)$ *for every* $x \in [0,1]$

(ii) $\widetilde{M}(\alpha - \beta) > 0$

(iii) there exist $k \in (0,1)$ such that the inequalities $\widetilde{M}(Tu - Tv) \le k\widetilde{M}(u - v) + (1 - k)d_{\widetilde{M}}$ holds for any $u \in A$, $v \in B$

(iv) for any $u \in A$ and $v \in B$ there hold the inclusions $Tu \in B$ and $Tv \in A$.

Then T is a cyclic $\widetilde{M}$–contraction and there exists a unique $u \in A$ such that u is a $\widetilde{M}$–best proximity point of T in A, $T^2u = u$ and for any $u_0 \in A$ the sequence $\{T^{2n}u_0\}_{n=1}^{\infty}$ is $\widetilde{M}$–convergent to u.

Example 7. *Let $L_{\widetilde{2}}[0,1]$ be the modular function space, which is generated by the Orlicz function $M(t) = |t|^2$. Let us consider the functions $f(x) = |x|$, $K(x,s) = \dfrac{xs}{2}$ and $g(x) = \dfrac{5}{6}x$. Denote the sets $A = \{u \in L_{\widetilde{2}}[0,1] : x \le u(x) \le nx\}$, $B = \{v \in L_{\widetilde{2}}[0,1] : -nx \le v(x) \le -x\}$, where $n \in \mathbb{N}$. The map $T : A \cup B \to A \cup B$, defined by*

$$Tu(x) = -\mathrm{sign}(u(x)) \left(g(x) + \int_0^1 K(x,s) f(u(s)) ds \right)$$

is a cyclic $\widetilde{M}$–contraction and $x \in A$ is a $\widetilde{M}$–best proximity point of T in A, $T(x) = -x$, $T^2(x) = T(-x) = x$.

Coupled fixed points and coupled best proximity points in modular function spaces

Definition 35. *Let A and B be nonempty subsets of a modular function space L_ρ, $F : A \times A \to B$. An ordered pair $(x, y) \in A \times A$ is called a coupled best proximity point of F in A if $\rho(x - F(x, y)) = \rho(y - F(y, x)) = d$.*

Definition 36. *Let A be nonempty subset of a modular function space X, $F : A \times A \to A$. An ordered pair $(x, y) \in A \times A$ is said to be a coupled fixed point of F A if $x = F(x, y)$ and $y = F(y, x)$.*

It is easy to see that if $A = B$ in Definition 35, then a coupled best proximity point reduces to a coupled fixed point.

Definition 37. *([49])Let A and B be nonempty subsets of a modular function space L_ρ. Let $F : A \times A \to B$ and $G : B \times B \to A$. For any pair $(x, y) \in A \times A$ we define the sequences $\{x_n\}_{n=0}^\infty$ and $\{y_n\}_{n=0}^\infty$ by $x_0 = x$, $y_0 = y$ and*

$$\begin{array}{ll} x_{2n+1} = F(x_{2n}, y_{2n}), & y_{2n+1} = F(y_{2n}, x_{2n}) \\ x_{2n+2} = G(x_{2n+1}, y_{2n+1}), & y_{2n+2} = G(y_{2n+1}, x_{2n+1}) \end{array}$$

for all $n \geq 0$.

Everywhere in this section, when considering the iterated sequences $\{x_n\}_{n=0}^\infty$, $\{y_n\}_{n=0}^\infty$ we will assume that they are the sequences defined in Definition 37.

Coupled fixed points for ρ–contraction maps in modular function spaces

Definition 38. *Let A be nonempty subsets of a modular function space L_ρ, $F : A \times A \to A$ is said to be a ρ–contraction if there exist non-negative numbers α, β, such that $\alpha + \beta < 1$ and there holds the inequality $\rho(F(x, y) - F(u, v)) \leq \alpha\rho(x - u) + \beta\rho(y - v)$ for all $x, y, u, v \in A$.*

Theorem 18. *Let $\rho \in \mathfrak{R}$. Let $A \subset L_\rho$ be nonempty, ρ–closed and ρ–bounded. Let $F : A \times A \to A$ be a ρ–contraction. Then F has unique coupled fixed points $(x, y) \in A$. Moreover for any $(x_0, y_0) \in A$ the sequences $\{x_n\}_{n=0}^\infty$, $\{y_n\}_{n=0}^\infty$ converge to the unique coupled fixed points $(x, y) \in A$.*

Coupled best proximity points for cyclic ρ–contractions in modular function spaces

Definition 39. *Let A and B be nonempty subsets of a modular function space L_ρ, $F : A \times A \to B$ and $G : B \times B \to A$. The ordered pair (F, G) is said to be a cyclic ρ–contraction if there exist non-negative numbers α, β, such that $\alpha + \beta < 1$ and there holds the inequality $\rho(F(x, y) - G(u, v)) \leq \alpha\rho(x - u) + \beta\rho(y - v) + (1 - (\alpha + \beta))d_\rho(A, B)$ for all $(x, y) \in A \times A$ and $(u, v) \in B \times B$.*

Theorem 19. *Let $\rho \in \mathfrak{R}$. Assume that ρ satisfies (UC1), has the Δ_2–property and be uniformly continuous. Let $A, B \subseteq L_\rho$ be ρ–closed, ρ–bounded, convex subsets and $F : A \times A \to B$ and $G : B \times B \to A$ be an order pair (F, G) be cyclic ρ–contraction. Then there exists a* *unique order pair $(x,y) \in A \times A$ such that (x,y) is a coupled ρ–best proximity point of F in A (i.e. $\rho(x - F(x,y)) + \rho(y - F(y,x)) = 2d_\rho(A,B)$). There holds $x = G(F(x,y), F(y,x))$, $y = G(F(y,x), F(x,y))$, the order pair $(F(y,x), F(x,y))$ is a coupled ρ–best proximity point of G in B. More over for any initial guess $(x_0, y_0) \in A \times A$ the iterated sequences $\{x_n\}$, $\{y_n\}$ satisfied $\lim\limits_{n\to\infty} \rho(x_{2n} - x) = 0$, $\lim\limits_{n\to\infty} \rho(y_{2n} - y) = 0$, $\lim\limits_{n\to\infty} \rho(x_{2n+1} - F(x,y)) = 0$, $\lim\limits_{n\to\infty} \rho(y_{2n+1} - F(y,x)) = 0$.*

An Application of coupled best proximity points in modular function spaces

Let $p \in [1, +\infty)$, $a > 0$, $\alpha, \beta \in (0,1)$ be such that $\alpha + \beta < 1$ and $(1 - \alpha - \beta)a = \gamma$. Let us consider the system of equations

$$\left|\begin{array}{l} |(1+\alpha)x + \beta y + \gamma|^p = (2a)^p \\ |\alpha x + (1+\beta)y + \gamma|^p = (2a)^p \\ x \ge 0,\ y \ge 0. \end{array}\right. \tag{9}$$

It is easy to check, by using a Computer Algebra Software, that the ordered pair (a,a) is a solution of (9) if $p \in \mathbb{N}$. If we try to solve this system for $p \notin \mathbb{N}$ then the computer will give no answer.

Let us consider the space $\mathbb{R}_{|.|^p}$ of all reals endowed with the function modular $\rho_p(\cdot) = |\cdot|^p$. For $p = 1$ we get that the space $\mathbb{R}_{|\cdot|^1}$, which is a normed space and from [29] it follows that it is uniformly convex. Consequently if considered as modular function space we get that ρ_1 satisfies (UC1), has the Δ_2–property and is uniformly continuous and we can apply Theorem 19 in $\mathbb{R}_p$.

Let us consider the subsets $A = [a,b]$, $B = [-b,-a]$ for $0 < a < b$ of $\mathbb{R}_p$. Let us define the functions $F(x,y) = -\alpha x - \beta y - \gamma$ and $G(x,y) = -\alpha x - \beta y + \gamma$. The ordered pair (F,G) is an cyclic ρ–contraction ordered pair and from Theorem 19 it follows that there exists a unique order pair $(x,y) \in A \times A$ such that (x,y) is a coupled ρ–best proximity point of F in A (i.e. $\rho_p(x - F(x,y)) = (2a)^p$ and $\rho_p(y - F(y,x)) = (2a)^p$, which is just (9)). The solution can be approximated by using the sequence of consecutive iterations $\{x_n\}_{n=1}^\infty$ and $\{x_n\}_{n=1}^\infty$, defined in Definition 37, starting with an arbitrary guess points x_0 and y_0.

If we put $\alpha = \beta = \frac{1}{4}$, $\gamma = \frac{1}{2}$, $a = 1$, $b = 2$ and $p = 1$ we get Example 3.

Coupled fixed points for ρ–Kannan contractions in modular function spaces

Definition 40. *Let A be nonempty subsets of a modular function space L_ρ, $F : A \times A \to A$ is said to be a ρ–Kannan contraction if there exist $\alpha \in [0, 1/2)$, so that there holds the inequality $\rho(F(x,y) - F(u,v)) \le \alpha\,(\rho(x - F(x,y)) + \rho(u - F(u,v)))$ for all $x,y,u,v \in A$.*

Theorem 20. *Let $\rho \in \mathfrak{R}$. Let $A \subset L_\rho$ be nonempty, ρ–closed and ρ–bounded. Let $F : A \times A \to A$ be a cyclic ρ-Kannan contraction map. Then F has unique coupled fixed points $(x,y) \in A$. Moreover for any $(x_0,y_0) \in A$ the sequences $\{x_n\}$, $\{y_n\}$ defined by $x_1 = F(x_0,y_0)$, $y_1 = F(y_0,x_0)$, $x_{n+1} = F(x_n,y_n)$, $y_{n+1} = F(y_n,x_n)$ for $n = 1,2,\ldots$ are ρ-converge to the unique coupled fixed points $(x,y) \in A$, i.e. $\lim_{n\to\infty} \rho(x_n - x) = 0$ and $\lim_{n\to\infty} \rho(y_n - y) = 0$.*

Coupled best proximity points for cyclic ρ–Kannan contractions in modular function spaces

Definition 41. *Let A and B be nonempty subsets of the modular function space L_ρ. The pair of maps $F : A \times A \to B$ and $G : B \times B \to A$ is called an order cyclic ρ-Kannan contraction pair, if there is $\alpha \in [0, 1/2)$, so that there holds the inequality*

$$\rho(F(x,y) - G(u,v)) \leq \alpha\left(\rho(x - F(x,y)) + \rho(u - G(u,v))\right) + (1-2\alpha)d_\rho(A,B)$$

for any $x, y \in A$ and $u, v \in B$.

Theorem 21. *Let $\rho \in \mathfrak{R}$. Assume that ρ satisfies (UC1), has the Δ_2–property and be uniformly continuous. Let $A, B \subseteq L_\rho$ be ρ–closed, ρ–bounded, convex subsets and $F : A \times A \to B$ and $G : B \times B \to A$ be an order cyclic ρ-Kannan contraction pair (F, G). Then there exists a unique order pair $(x, y) \in A \times A$ such that (x, y) is a coupled ρ–best proximity point of F in A (i.e. $\rho(x - F(x, y)) + \rho(y - F(y, x)) = 2d_\rho(A, B)$). There holds $x = G(F(x, y), F(y, x))$, $y = G(F(y, x), F(x, y))$ the order pair $(F(y, x), F(x, y))$ is a coupled ρ–best proximity point of G in B. More over for any initial guess $(x_0, y_0) \in A \times A$ the iterated sequences $\{x_n\}_{n=0}^\infty$, $\{y_n\}_{n=0}^\infty$ satisfied $\lim\limits_{n\to\infty} \rho(x_{2n} - x) = 0$, $\lim\limits_{n\to\infty} \rho(y_{2n} - y) = 0$, $\lim\limits_{n\to\infty} \rho(x_{2n+1} - F(x, y)) = 0$, $\lim\limits_{n\to\infty} \rho(y_{2n+1} - F(y, x)) = 0$.*

Chapter IV
APPLICATIONS OF COUPLED FIXED POINTS AND COUPLED BEST PROXIMITY POINTS OF SEMI–CYCLIC MAPS IN THE INVESTIGATION OF EQUILIBRIUM IN DUOPOLY MARKETS

It may turn out difficult or impossible to solve (1), thus it is often advised to search for an approximate solution. Another drawback, when searching for an approximate solution is that it may not be stable.

We can present the solutions of (1) by an implicit formula, that defines the response functions, i.e. $x = \frac{c_1'(x) - P(x+y)}{P'(x+y)} = F(x, y)$ and $y = \frac{c_2'(y) - P(x+y)}{P'(x+y)} = f(x, y)$.

Thus, finding the equilibrium production (x, y) can be considered as the problem of finding coupled fixed points. Following [24], we will call the functions F and f the response functions for the two manufacturers participating in the duopoly market.

In real models, it is possible to construct response functions that are not a consequence of the problem of maximizing participants' payoff functions Π_i, $i = 1, 2$. In reality, each participant reacts (i.e. has a response function) according to the production he has sold in the previous interval and the quantity of goods sold by his competitor, taking into account the demand function. For example, let n currently the quantities of goods produced by the two participants be the ordered pair (x_n, y_n). Let producer one change his production for the moment $n + 1$ to $x_{n+1} = F(x_n, y_n)$, and let producer two change his production to $y_{n+1} = f(x_n), y_n)$. In this way an iterative series of productions $\{(x_n, y_n)\}_{n=1}^\infty$ is generated. We will have equilibrium in the market if there are two possible quantities of output x and y, so that $x = F(x, y)$ and $y = f(x, y)$.

The problem of solving the equations $x = F(x, y)$ and $y = f(x, y)$ is the problem of finding coupled fixed points for an ordered pair of maps (F, f) [11] (or more precisely of the problem of modified coupled fixed points). Using the model through response functions, we change the payoff maximization problem to a coupled fixed points one and thus we can get rid of the conditions for convexity and differentiability [1, 25, 37].

Focusing on response functions, allows to put together Cournot and Bertand models. Indeed let the first company have reaction be $F(X, Y)$ and the second one $f(X, Y)$, where $X = (x, p)$ and $Y = (y, q)$. Here x and y denote the output quantity and (p, q) are the prices set by players. In this case companies can compete in terms of both price and quantity.

Semi–cyclic maps

In order to apply the technique of coupled best proximity points and coupled fixed points we will generalize the mentioned above notions. When we investigate duopoly with players' response functions F and f, we have seen that each player using the information about his production and the rival's production choose a change in his production, i.e., we define $F : A \times B \to A$, $f : A \times B \to B$ instead of the cyclic type of maps (Definition 8). The maps are not noncyclic maps ($F(A) \subseteq A$ and $f(B) \subseteq B$ [23]). Therefore we introduce the notion of semi-cylcic maps.

Definition 42. *Let A, B be nonempty subsets of X. The ordered pair of maps $F : A \times B \to A$, $f : A \times B \to B$ will be called a semi–cyclic ordered pair of maps.*

Definition 43. *Let A, B be nonempty subsets of a metric space (X, ρ) and (F, f), $F : A \times B \to A$, $f : A \times B \to$ be a semi–cyclic ordered pair of maps. For any pair $(x, y) \in A \times B$ we define the sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ by $x_0 = x$, $y_0 = y$ and $x_{n+1} = F(x_n, y_n)$, $y_{n+1} = f(x_n, y_n)$ for all $n \geq 0$.*

Coupled fixed points for semi–cyclic maps

We generalize the notion of coupled fixed points for semi–cyclic maps.

Definition 44. *Let A, B be nonempty subsets of a metric space (X, ρ) and (F, f), $F : A \times B \to A$, $f : A \times B \to$ be a semi–cyclic ordered pair of maps. An ordered pair $(\xi, \eta) \in A \times B$ is called a coupled fixed point of (F, f) if $\xi = F(\xi, \eta)$ and $\eta = f(\xi, \eta)$.*

If $B = A$ and $f(x, y) = F(y, x)$, then we get the definition of a coupled fixed point from [11].

We will generalize the contractive condition from [22] for semi-cyclic maps.

Definition 45. *Let A, B be nonempty subsets of a metric space (X, ρ) and (F, f), $F : A \times B \to A$, $f : A \times B \to$ be a semi–cyclic ordered pair of maps. Let there exist a subset $D \subseteq A \times B$, such that $F : D \to A$, $f : D \to B$ and $(F(x, y), f(x, y)) \subseteq D$ for every $(x, y) \in D$. The semi–cyclic ordered pair of maps (F, f) is said to be a contraction of type one semi–cyclic ordered pair if there exist non-negative numbers $\alpha, \beta, \gamma, \delta$, such that $\max\{\alpha + \gamma, \beta + \delta\} < 1$ and there holds the inequality*

$$\rho(F(x, y), F(u, v)) + \rho(f(z, w), f(t, s) \leq \alpha\rho(x, u) + \beta\rho(y, v) + \gamma\rho(z, t) + \delta\rho(w, s) \tag{10}$$

for all $(x, y), (u, v), (z, w), (t, s) \in D$.

Theorem 22. *Let A, B be nonempty subsets of a metric space (X, ρ). Let there exist a closed subset $D \subseteq A_x \times A_y$ and maps $F : D \to A_x$ and $f : D \to A_y$, such that $(F(x, y), f(x, y)) \subseteq D$ for every $(x, y) \in D$. Let the ordered pair (F, f) be a semi–cyclic contraction of type one. Then*

(i) there exists a unique pair (ξ, η) in D, which is a unique coupled fixed point for the ordered pair (F, f). Moreover the iteration sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$, defined in Definition 43 converge to ξ and η respectively, for any arbitrary chosen initial guess $(x, y) \in A_x \times A_y$

(ii) a priori error estimates hold $\max\{\rho(x_n, \xi), \rho(y_n, \eta)\} \leq \frac{k^n}{1-k}(\rho(x_1, x_0) + \rho(y_1, y_0))$

(iii) a posteriori error estimates hold $\max\{\rho(x_n, \xi), \rho(y_n, \eta)\} \leq \frac{k}{1-k}(\rho(x_{n-1}, x_n) + \rho(y_{n-1}, y_n))$

(iv) rate of convergence for the sequences of successive iterations

$$\rho(x_n, \xi) + \rho(y_n, \eta) \leq k\left(\rho(x_{n-1}, \xi) + (y_{n-1}, \eta)\right),$$

where $k = \max\{\alpha + \gamma, \beta + \delta\}$.

If in addition $f(x, y) = F(y, x)$ then the coupled fixed point (x, y) satisfies $x = y$.

Applications of Theorem 22 and examples

Everywhere we will assume that competing companies in the market produce goods that are, if not identical, perfect substitutes. Although the two goods may have nothing in common, the assumption of being perfect substitutes means that within the each type of goods customers are free to replace a product from the first company with one produced by the second and vice versa.

We will state Theorem 22 in an economic language.

Assumption 1. *Let there be a duopoly market, satisfying the following assumptions:*

1. *the two firms are producing homogeneous goods that are perfect substitutes*
2. *the first firm can produce qualities from the set A and the second firm can produce qualities from the set B, where A and B be closed, nonempty subsets of a complete metric space (X, ρ)*
3. *let there exist a closed subset $D \subseteq A \times B$ and maps $F : D \to A$ and $f : D \to B$, such that $(F(x, y), f(x, y)) \subseteq D$ for every $(x, y) \in D$, be the response functions for firm one and two respectively*
4. *let the semi–cyclic ordered pair (F, f) satisfies the conditions of Theorem 22.*

Then there exists a unique pair (ξ, η) in D, such that $\xi = F(\xi, \eta)$ and $\eta = f(\xi, \eta)$, i.e. a market equilibrium pair. Moreover the iteration sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$, defined in Definition 43 converge to ξ and η respectively and the error estimates of Theorem 22 hold.

If in addition $f(x.y) = F(y, x)$ then the coupled fixed point (x, y) satisfies $x = y$.

Remark 1. *Let the two players have one and the same response function. That is if player one has a production x and player two has a production y then the first player reaction will be $F(x,y)$ and the second player reaction will be $f(x,y) = F(y,x)$. It follows that the equilibrium pair (x,y) will satisfy $x = y$, i.e. both firms will have equal production. This means that if both firms have one and the same technology, one and the same knowledge on the market that will affect to one and the same response functions, then the equilibrium will be reached at the level of equal productions.*

Remark 2. *Let us consider a duopoly market. Let the two firms produce qualities from the set A and the second firm can produce qualities from the set B, where A and B are nonempty subsets of a complete metric space (X,ρ). Any of the firms can produce a bundle of products $x = (x_1, x_2, \dots x_n) \in X$. Assumption 1 ensures the existence and uniqueness of the production bundles $(x_1, x_2, \dots x_n), (y_1, y_2, \dots y_n) \in A \times B$ of n-goods, that present the equilibrium in a duopoly economy.*

A Linear Case, When Each Player Is Producing a Single Product, Goods Being Perfect Substitutes

Example 8. *Let us consider a market with two competing firms, each firm producing just one product, and both goods are perfect substitutes. Let the two firms produce quantities $x \in A$ and $y \in B$, respectively, where $A, B \subset [0, +\infty)$ and (X,ρ) be the complete metric space $(\mathbb{R}, |\cdot|)$. Let us consider the response functions of player one be $F(x,y) = a - s - px - qy$ and player two be $f(x,y) = a - r - \mu y - \nu x$, where*

1. $a, s, r, p, q, \mu, \nu > 0$, $s < a$, $r < a$, $\max\{p + \mu, q + \nu\} < 1$
2. $A = \left[0, \frac{a-s}{p}\right] \cap \left[0, \frac{a-r}{\mu}\right]$ *and* $B = \left[0, \frac{a-s}{q}\right] \cap \left[0, \frac{a-r}{\nu}\right]$
3. *D can be defined in three ways:*

   (3a) $D = \left[0, \frac{a\mu - aq - s\mu - qr}{\mu p - \nu q}\right] \times \left[0, \frac{ap - a\nu + s\nu - pr}{\mu p - \nu q}\right]$, *provided that*

   $$a - s \leq \frac{a\mu - aq - s\mu - qr}{\mu p - \nu q} \quad \text{and} \quad a - r \leq \frac{ap - a\nu + s\nu - pr}{\mu p - \nu q}$$

   (3b) $D = [0, a - s] \times [0, a - r]$, *provided that* $\mu r + \nu s - a\mu - a\nu + a - r > 0$ *and* $ps + qr - ap - aq + a - s > 0$

   (3c) $D = \begin{cases} 0 \leq x \leq \dfrac{a-s}{p} \\ 0 \leq y \leq \dfrac{a - r - \mu x}{\nu}. \end{cases}$

Example 9. *Let us consider a classic example, where the price function is linear and so are the cost functions of both players. Assuming the feasible market price is defined by $P(x,y) = 120 - x - y$, it is expected that additional output x from the first company as well as extra production y of the second one will cause a decrease in prices. Therefore under equilibrium conditions $x + y$ will be the total production of the two firms and it will also be reflected in prices. Let the two firms have cost functions equal to $30x$ and $20y$, respectively.*

Following the Cournot model after solving (1) we get the response functions $F : D \to A$ and $f : D \to B$ of the two firms $F(y) = \frac{90-y}{2}$ and $f(x) = \frac{100-x}{2}$, where $B = [0, 90]$, $A = [0, 100]$ and $D = A \times B$. Consequently it is a special case of the general example with $a = 60$, $s = 15$, $r = 10$, $p = 0$, $q = \frac{1}{2}$, $\mu = \frac{1}{2}$, $\nu = 0$.

Table 8. Values of the iterated sequence (x_n, y_n) if started with $(40, 60)$.

n	0	1	2	5	10	20
x_n	40	15	30.0	25.94	26.68	26.67
y_n	60	30	42.5	36.25	36.69	36.67

Table 9. Values of the iterated sequence (x_n, y_n) if started with $(100, 20)$.

n	0	1	2	5	10	20
x_n	100	35	45.0	27.19	26.74	26.67
y_n	20	0	32.5	34.38	36.65	36.67

Table 10. Number n of iterations needed by the a priori estimate if started with $(100, 20)$.

ε	0.1	0.01	0.001	0.0001	0.00001
n	11	15	18	21	25

Table 11. Number n of iterations needed by the a posteriori estimate if started with $(100, 20)$.

ε	0.1	0.01	0.001	0.0001	0.00001
n	6	8	11	14	17

A nonlinear case, when each player is producing a single product, while goods sold are perfect substitutes

Example 10. *Let us consider a market with two competing firms, producing perfect substitute products with quantities $x \in A$ and $y \in B$, respectively, where $A, B \subset [0, +\infty)$ and (X, ρ) is the complete metric space $(\mathbb{R}, |\cdot|)$. Let us assume that each firm produces at least one item, i.e., $x, y \geq 1$. Let us consider the response functions of player one $F(x, y) = \frac{90-x-\frac{y}{8}-\frac{\sqrt{y}}{2}}{2}$ and player two $f(x, y) = \frac{100-\frac{x}{4}-y-\sqrt{x}}{3}$, where*

(3a) $A = [1, 44]$ *and* $B = [1, 33]$

(3b) D *can be defined as* $D = A \times B$.

There exists an equilibrium pair (x, y) and for any initial start in the economy the iterated sequences (x_n, y_n) converge to the market equilibrium (x, y). We get in this case that the equilibrium pair of the production of the two firms is $(28.3, 21.9)$ and the total production will be $a = 50.2$.

Each player is producing two product types, goods from each type being perfect substitutes

Example 11. *Let us consider a market with two competing firms, and each firm is producing two product types. For simplicity we assume that goods from each type produced by major players are perfect substitutes. While it is possible that two types have nothing it common, it still means that within each type customers can freely replace a product from the first company with one manufactured by the second one. Let us assume that each firm produces at least one item from each product, i.e.* $x = (x_1, x_2),\quad y = (y_1, y_2),\quad x_1, x_2, y_1, y_2 \geq 1$. *Let us denote the production of the two players by* $x = (x_1, x_2)$ *and* $y = (y_1, y_2)$*, respectively.*

Let the market of the two goods be endowed with the p *norm,* $p \in [1, \infty)$*, i.e.,*

$$\rho((x_1, x_2), (y_1, y_2)) = \|(x_1, x_2) - (y_1, y_2)\|_p = \left(|x_1 - y_1|^p + |x_2 - y_2|^p\right)^{1/p}.$$

Let us consider the response functions

$$F(x, y) = (F_1(x, y), F_2(x, y)) \quad \text{and} \quad f(x, y) = (f_1(x, y), f_2(x, y))$$

defined by

$$F(x, y) = \begin{cases} \dfrac{90 - \dfrac{x_1 + x_2}{2} - \dfrac{y_1 + y_2}{3}}{3}, \\ \dfrac{90 - \dfrac{x_1 + x_2}{2} - \dfrac{y_1 + y_2}{3}}{3}; \end{cases} \quad f(x, y) = \begin{cases} \dfrac{100 - \dfrac{x_1 + x_2}{4} - \dfrac{y_1 + y_2}{3}}{4} \\ \dfrac{100 - \dfrac{x_1 + x_2}{4} - \dfrac{y_1 + y_2}{3}}{4}. \end{cases}$$

where

(1) $A = [0, 30] \times [0, 30]$ *and* $B = [0, 25] \times [0, 25]$

(2) $D = [0, 30] \times [0, 30] \times [0, 25] \times [0, 25]$

There exists an equilibrium pair (x, y) and for any initial start in the economy the iterated sequences (x_n, y_n) converge to the market equilibrium (x, y). We get in this case that the equilibrium pair of the production of the two firms is $x = (19.27, 19.27)$, $y = (19.36, 19.36)$ and the total production will be $a = (38.63, 38.63)$.

The players are producing a single product and compete on both quantities and prices

There is a large number of goods where companies can compete on both quantities and prices. In this case the equilibrium would depend on a balanced decision on what market share to target at a reasonable price. Let us assume that there are only two major players that produce homogeneous products. The first company can produce quantities from the set $A \subseteq [0, \infty)$ at a price $p \in P \subseteq [0, \infty)$ and the second one can produce quantities from the set $B \subseteq [0, \infty)$ at a price $p \in Q \subseteq [0, \infty)$, where A, B, P, Q are nonempty subsets. Let $A \times P$, $B \times Q$ be subsets of a complete metric space $(\mathbb{R}^2, \rho)$.

Assumption 2. *Let there be a duopoly market, satisfying the assumptions:*

1. *the two firms are producing homogeneous, perfect substitute products*
2. *The first firm can produce quantities from the set A at a price $p \in P$ and the second firm can produce quantities from the set B at a price $p \in Q$, where $A \times P$, $B \times Q$ are nonempty, closed subsets of a complete metric space $(\mathbb{R}^2, \rho)$*
3. *let there exist a closed subset $D \subseteq A \times P \times B \times Q$, such that $F : D \to A \times P$, $f : D \to B \times Q$ and $(F(x,p,y,q), f(x,p,y,q)) \subseteq D$ for every $(x,p,y,q) \in D$ be the response functions for firm one and two respectively*
4. *let there exist $\alpha, \beta, \gamma, \delta > 0$, $\max\{\alpha+\gamma, \beta+\delta\} < 1$, such that the inequality*
$$\rho(F(X,Y),F(U,V))+\rho(f(Z,W),f(T,S)) \leq \alpha\rho(X,U)+\beta\rho(Y,V)+\gamma\rho(Z,T)+\delta\rho(W,S),$$
where we use the notations $X = (x,p_1)$, $Y = (y,q_1)$, $U = (u,p_2)$, $V = (v,q2)$, $Z = (z,p_3)$, $W = (w,q_3)$, $T = (t,p_4)$, $S = (s,q_4)$ holds for all
$$(x,p_1,y,q_1),(u,p_2,v,q_2),(z,p_3,w,q_3),(t,p_4,s,q_4) \in D.$$

Then there exists a unique pair (ξ, p, η, q) in $A \times P \times B \times Q$, which is a coupled fixed point for the semi–cyclic ordered pair of maps (F, f), i.e. a market equilibrium pair. Moreover the iteration sequences $\{x_n\}_{n=0}^{\infty}$, $\{p_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$, defined in Definition 43 converge to ξ, p, η, and q and the error estimates from Theorem 22 hold.

If in addition $f(X,Y) = F(Y,X)$ then the coupled fixed point (X,Y) satisfies $X = Y$, i.e. $x = y$ and $p = q$.

The considered model for simultaneous competition in prices and quantities can be called the Cornot-Bertrand model.

Example of a duopoly model, where players compete on quantities and prices simultaneously

Example 12. *Let us consider a market with two competing firms, producing the same product, and selling it at a price p and q respectively, i.e. $X = (x,p), Y = (y,q)$. Let us consider the response functions*
$$F(X,Y) = (F_1(X,Y), F_2(X,Y)) \quad \text{and} \quad f(X,Y) = (f_1(X,Y), f_2(X,Y))$$
defined by
$$F(X,Y) = \begin{cases} \dfrac{90 - \dfrac{x}{2} - \dfrac{y}{3}}{3}, \\ \dfrac{4 - \dfrac{p}{2} - \dfrac{q}{3}}{3}; \end{cases} \quad f(X,Y) = \begin{cases} \dfrac{100 - \dfrac{x}{4} - \dfrac{y}{3}}{4} \\ \dfrac{5 - \dfrac{p}{4} - \dfrac{q}{3}}{4}. \end{cases}$$
Let $X = (x,p)$ and $Y = (y,q)$ be subsets of $(\mathbb{R}^2, \|\cdot\|_2)$ (the two dimensional Euclidean space). Let

1. *$A \times P = [0,100] \times [0,5]$ and $B \times Q = [0,100] \times [0,4]$*

2. $D = [0, 100] \times [0, 5] \times [0, 100] \times [0, 4]$.

There exists an equilibrium pair (X, Y) and for any initial start in the economy the iterated sequences (X_n, Y_n) converge to the market equilibrium (X, Y). We get in this case that the equilibrium pair of the production of the two firms is $X = (23.64, 1.03)$, $Y = (21.71, 1.09)$.

Coupled fixed points for semi-cyclic Hardy–Rogers type of contraction maps

We will generalize the notions from [17], by considering two different metric spaces (Z_i, ρ_i), $i = 1, 2$.

Definition 46. *Let X_1, X_2 be nonempty subsets of the metric spaces (Z_1, ρ_1) and (Z_2, ρ_2), respectively, $F_i : X_1 \times X_2 \to X_i$ for $i = 1, 2$ be semi–cyclic ordered pair of maps. For any pair $(x, y) \in X_1 \times X_2$ we define the sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ by $x_0 = x$, $y_0 = y$ and $x_{n+1} = F_1(x_n, y_n)$, $y_{n+1} = F_2(x_n, y_n)$ for all $n \geq 0$.*

Definition 47. *Let X_1, X_2 be nonempty subsets of the metric spaces (Z_1, ρ_1) and (Z_1, ρ_2), respectively, $F_i : X_1 \times X_2 \to X_i$ for $i = 1, 2$. An ordered pair $(\xi, \eta) \in X_1 \times X_2$ is called a coupled fixed point of (F_1, F_2) if $\xi = F_1(\xi, \eta)$ and $\eta = F_2(\xi, \eta)$.*

Just to fit some of the formulas into the text field, let us denote

$$\begin{array}{rcl} M & = & M_{F_1,F_2}(x, y, u, v) \\ & = & \rho_1(x, F_1(x, y)) + \rho_2(y, F_2(x, y)) + \rho_1(u, F_1(u, v)) + \rho_2(v, F_2(u, v)) \end{array}$$

and

$$\begin{array}{rcl} N & = & N_{F_1,F_2}(x, y, u, v) \\ & = & \rho_1(x, F_1(u, v)) + \rho_2(y, F_2(u, v)) + \rho_1(u, F_1(x, y)) + \rho_2(v, F_2(x, y)). \end{array}$$

Theorem 23. *Let (X_1, ρ_1) and (X_2, ρ_2) be two complete metric spaces. Let there are two maps $F_i : X_1 \times X_2 \to X_i$, for $i = 1, 2$ and let there are non–negative constants k_i for $i = 1, 2, 3$, so that $k_1 + 2k_2 + 2k_3 < 1$ and the ordered pair of maps (F_1, F_2) satisfies the inequality*

$$\sum_{i=1}^{2} \rho_i(F_i(x, y), F_i(u, v)) \leq k_1(\rho_1(x, u) + \rho_2(y, v)) + k_2 M_{F_1,F_2}(X) + k_3 N_{F_1,F_2}(X) \tag{11}$$

for any $(x, y), (u, v) \in X_1 \times X_2$, and we have denoted $X = (x, y, u, v)$. Then

- *(i) there is a unique coupled fixed point $(\xi, \eta) \in X_1 \times X_2$ of (F_1, F_2) and moreover for any initial guess $(x_0, y_0) \in x$ the iterated sequences $x_n = F_1(x_{n-1}, y_{n-1})$ and $y_n = F_2(x_{n-1}, y_{n-1})$, for $n = 1, 2, \ldots$ converge to the coupled fixed point (ξ, η)*
- *(ii) there holds the a priori error estimates*

$$\rho_1(\xi, x_n) + \rho_2(\eta, y_n) \leq \frac{k^n}{1-k}(\rho_1(x_0, x_1) + \rho_2(y_0, y_1))$$

(iii) there a posteriori error estimate

$$\rho_1(\xi, x_n) + \rho_2(\eta, y_n) \leq \frac{k}{1-k}(\rho_1(x_n, x_{n-1}) + \rho_2(y_n, y_{n-1}))$$

(iv) the rate of convergence

$$\rho_1(\xi, x_n) + \rho_2(\eta, y_n) \leq k(\rho_1(\xi, x_{n-1}) + \rho_2(\eta, y_{n-1}))$$

where $k = \frac{k_1+k_2+k_3}{1-k_2-k_3}$.

If in addition $X_1, X_2 \subseteq X$, *where* (X, ρ) *is a complete metric space and* $F_2(x, y) = F_1(y, x)$, *then the coupled fixed point* (ξ, η) *satisfies* $\xi = \eta$.

Remark 3. *By* $F_2(x, y) = F_1(y, x)$, *actually we assume that* F_i *is defined on the set* $(X_1 \cup X_2) \times (X_1 \cup X_2)$. *It is possible* $F_1(x_1, x_2) \notin X_1$ *not to hold inequality (11), provided that* $x_1, x_2 \in X_1$ *and* $x_2 \notin X_2$. *Therefore in these case we should assume that* $X_1 \equiv X_2$.

Connection between the second order conditions and the contraction type conditions for semi–cyclic contraction pairs of maps

Let both participants have rational behavior, i.e. they want to maximize their payoff, assuming that the functions P and c_i, $i = 1, 2$ are differentiable, we reach with the system of equations (1).

The equilibrium pair (x_0, y_0) of production is a solution of (1) [24, 50]. To ensure that the solution (x_0, y_0) of (1) will present a maximization of the payoff functions a sufficient condition is that Π_i be concave or is satisfied the second order conditions [24, 50].

By using of response functions we alter the maximization problem into a coupled fixed point one thus all assumptions of concavity and differentiability can be skipped.

Let us consider Theorem 23, so that X_1, X_2 be nonempty closed subsets of the complete metric space (X, ρ), instead of being subsets of two different metric spaces, and constants $\beta = \gamma = 0$. By putting $u = x$ and $v = y$ in Theorem 22 we get

$$\begin{aligned} S_2 &= \rho(F_1(x, y), F_1(u, v)) + \rho(F_2(x, y), F_2(u, v) \\ &\leq \alpha\rho(x, u) + \beta\rho(y, v) + \gamma\rho(x, u) + \delta\rho(y, v) \leq s(\rho(x, u) + \rho(y, v)), \end{aligned} \tag{12}$$

where $s = \max\{\alpha + \gamma, \beta + \delta\} < 1$.

Consequently Theorem 22 is a corollary of Theorem 23.

Theorem 23 may be stated in an economic language for $k_2 = k_3 = 0$.

Assumption 3. *Let there is a duopoly market, satisfying the following assumptions:*

1. *the two firms are producing homogeneous goods that are perfect substitutes.*
2. *the first firm can produce quantities from the set* X_1 *and the second firm can produce quantities from the set* X_2, *where* X_1 *and* X_2 *be closed, nonempty subsets of a complete metric space* (X, ρ)
3. *let there exist a closed subset* $D \subseteq X_1 \times X_2$ *and maps* $F_i : D \to X_i$, *such that* $(F_1(x, y), F_2(x, y)) \subseteq D$ *for every* $(x, y) \in D$, *be the response functions for firm one and two respectively*

4. let there exist $\alpha < 1$, such that the inequality

$$\rho(F_1(x,y),F_1(u,v)) + \rho(F_2(x,y),F_2(u,v)) \leq \alpha(\rho(x,u) + \rho(y,v)) \tag{13}$$

holds for all $(x,y),(u,v) \in X_1 \times X_2$.

Then there exists a unique pair (ξ,η) in D, such that $\xi = F_1(\xi,\eta)$ and $\eta = F_2(\xi,\eta)$, i.e. a market equilibrium pair.

If in addition $F_2(x,y) = F_1(y,x)$ then the coupled fixed point (ξ,η) satisfies $\xi = \eta$.

In addition to providing sufficient conditions for the existence of market equilibrium, Assumption 2 and Assumption 3 also provide sufficient conditions for the stability of the sewuence of successive responses over time of the two participants. Of course, we assume that over time, companies do not change their response functions.

It has been proved that from (13) follows (2), i.e. if the response functions satisfy (13), being obtained after differentiating the payoff functions, then they have the corresponding partial derivatives and the second-order condition for the coupled fixed point is satisfied.

Example 13. *Let us consider a model of a duopoly market with a price function $P(x,y) = 100 - x - y$ and cost functions $C_1(x) = \frac{x^2}{2}$ and $C_2(y) = \frac{y^2}{2}$.*

By (11) we get

$$\left| \begin{array}{l} \frac{\partial \Pi_1(x,y)}{\partial x} = 100 - 3x - y = 0 \\ \frac{\partial \Pi_2(x,y)}{\partial y} = 100 - x - 3y = 0. \end{array} \right. \tag{14}$$

The second order conditions are $\frac{\partial^2 \Pi_1(x,y)}{\partial x^2} = -3 < 0$ and $\frac{\partial^2 \Pi_2(x,y)}{\partial y^2} = -3 < 0$. Therefore the solution of (14) is market equilibrium, because there holds (2). Unfortunately the response functions in the consedered model are $F(x,y) = 100 - 2x - y$ and $f(x,y) = 100 - x - 2y$ and they do not satisfy Assumption 2.

Table 12. Values of the iterated sequence (x_n, y_n) if stared with $(20, 30)$.

n	0	1	2	...	2k	2k+1
x_n	20	30	20	...	20	30
y_n	30	20	30	...	30	20

Table 13. Values of the iterated sequence (x_n, y_n) if stared with $(20, 31)$.

n	0	1	2	3	4	5	6
x_n	20	29	24	17	60	0	100
y_n	31	18	35	6	71	0	100

In both cases Table 12 and Table 13 we see that the process is not converging.

Let us point out that the system (11) may have more than one solution (x,y), satisfying the second order conditions (2). In this case we will need further investigation to find which one of the solutions is the solution of the optimization problem of Cournot's model.

Therefore although the considered has a stronger restriction than (11), it is a different one from the well known payoff maximization Cournot's model

Comments on the coefficients α, β, γ and δ in Theorem 22

Although the Theorem 22 is a consequence of Theorem 23 it seems that the usage of four coefficients may give better understanding of duopoly markets.

Let the response functions F_1 and F_2 satisfy

$$\rho(F_1(x,y),F_1(u,v)) \le \alpha\rho(x,u) + \beta\rho(y,v) \tag{15}$$

and

$$\rho(F_2(x,y),F_2(u,v)) \le \gamma\rho(x,u) + \delta\rho(y,v). \tag{16}$$

If $\max\{\alpha+\gamma, \beta+\delta\} \in (0,1)$, then by summing up (15) and (16) the model satisfies inequality (10). Let us assume that α and δ are close to 1 and β and γ are close to 0. This means that both players do not pay too much attention to the behavior of the production of the other one. They are interested mostly of their own productions.

Example 14. *Let us consider a model with the following response functions $F_1(x,y) = 45 - 0.98x - 0.09y$ and $F_2(x,y) = 50 - 0.01x - 0.9y$. An example of Cournot model can be considered $P(x,y) = 50 - 0.09x - 0.01y$, and cost functions $C_1(x) = 0.985x^2$ and $C_2(y) = 0.86y^2$.*

Thus we get that

$$|x_{n+2} - x_{n+1}| = |F_1(x_n,y_n) - F_1(x_{n+1},y_{n+1})| \le 0.98|x_n - x_{n+1}| + 0.09|y_n - y_{n+1}|$$

and

$$|y_{n+2} - y_{n+1}| = |F_2(x_n,y_n) - F_2(x_{n+1},y_{n+1})| \le 0.01|x_n - x_{n+1}| + 0.9|y_n - y_{n+1}|,$$

which can be interpenetrated as any player take account just on his change of the production. The market equilibrium is $(24.06, 26.18)$.

Table 14. Values of the iterated sequence (x_n, y_n) if stared with $(10, 30)$.

n	0	1	2	3	10	21	50	120	121	599	600
x_n	10	37	12	35	16.8	30.8	21.1	22.64	25.43	24.07	24.05
y_n	30	18	33	20	28.6	24.1	25.8	26.03	26.34	26.19	26.18

We see from the Table 14 that at the very beginning the osculations of the sequence of productions are big and it take a lot of time to get close enough to the equilibrium values.

An application of Theorem 23 in a newly investigated duopoly model

A deep analysis of a class of oligopoly markets is presented in [3]. In section 2 in [3] authors analyze market equilibrium, obtained by the use of the first and second order conditions. They have assumed $P(Q) = Q^{-1/\mu}$, where P be the market price, $x, y \ge 0$ are the quantity supplied by firm one and two, respectively, $Q = x + y$ be the total output and $\mu > 0$ be a parameter. Both players share a linear cost function with constant average and marginal cost $c_i > 0$. As far as part of the results in [3] are for $c_i = c$ for $i = 1, 2$, let us assume that $c_1 = c_2 = c$. The first order conditions in [3] yield to the system of equations:

$$\left| \begin{array}{l} x = \mu Q - c\mu Q^{1+\frac{1}{\mu}} \\ y = \mu Q - c\mu Q^{1+\frac{1}{\mu}}. \end{array} \right.$$

Both players share one and the same response function

$$F_1(x, y) = F_2(x, y) = F(x, y) = \mu Q - c\mu Q^{1+\frac{1}{\mu}},$$

where $Q = x + y$. The inequality $0 < cQ_{\max}^{1/\mu} < \frac{1-2\mu}{2(1+\mu)} < 1$ holds true for any $\mu \in [0, 1/2)$.

The analysis in [3], using of the second order conditions yields that there exists a market equilibrium if $\mu \geq \max\left\{1, \frac{x}{2y-x}, \frac{y}{2x-y}\right\}$ and it can not be said anything if $\mu < 1$.

Thus Theorem 23 covers and cases that are not covered by the classical first and second order conditions.

Applications of Theorem 23 for optimization of non–differentiable payoff functions and examples

It seems from Theorem 23 that we can impose different type of contraction conditions that will be not equivalent to (13). We can restate Theorem 23, when $k_1 = k_3 = 0$ in the economic language.

Example 15. *Let us consider a market with two competing firms, producing perfect substitute products. Let us consider the response functions of player one and two be*

$$F_1(x, y) = \begin{cases} 0.2 & x \in [0, 0.8] \\ 0.1 & x \in (0.8, 1] \end{cases} \quad F_2(x, y) = \begin{cases} 0.9 & y \in [0, 0.1] \\ 0.8 & y \in (0.1, 1], \end{cases}$$

respectively.

There exists an equilibrium pair (x, y) and for any initial start in the economy the iterated sequences (x_n, y_n) converge to the market equilibrium (x, y). We get in this case that the equilibrium pair of the production of the two firms is $(0.8, 0.1)$.

The considered model with response functions F_1 and F_2 does not satisfies Theorem 22.

The example shows that if F_i were obtained by solving the optimization of the payoff functions, then we can not speak about the second order conditions as far as F_i are not differentiable.

A generalized response function

Looking at the models of the duopoly in the paragraphs so far, we did not pay attention to the products that were produced but not sold on the market. We assumed that in their response the companies take into account only the products sold on the market. In order for the constructed model to be more realistic, we must also take into account the redundant (or unrealized) production.

We assume that each market participant takes into account not only how much production it has sold on the market, but also how much is left in its warehouses (i.e it has produced without being able to sell it). We assume that each participant has complete information not only how much he has sold on the market, but also the amount sold by his competitor. Of course, none of the participants has information about the unrealized quantities of the competitor.

Let us denote the set of the possible productions of player i by U_i; the set of the realized production on the market by $P_i \subseteq U_i$; the set of its surplus quantities by s_i, $i = 1,2$. Let us put $X_i = P_i \times s_i$. Each of the players is not able to know the surplus production of the other one. Therefore a more real model of the response functions of the two player will be $f_1 : X_1 \times P_2 \to U_1$, $f_2 : X_2 \times P_1 \to U_2$. Starting at a moment t_0 with realized on the market productions $p_i^{(0)}$, surpluses $s_i^{(0)}$ and productions $u_i^{(0)}$, $i = 1,2$ for both players it results to a new productions of the players

$$u_1^{(1)} = f_1\left(p_1^{(0)}, p_2^{(0)}, s_1^{(0)}\right) \in U_i, \quad u_2^{(1)} = f_2\left(p_1^{(0)}, p_2^{(0)}, s_2^{(0)}\right) \in U_2.$$

The market reacts to this new levels of production by generating new surplus quantities $s_i^{(1)} = Q_i(u_1^{(1)}, u_2^{(1)})$, where $Q_i : U_1 \times U_2 \to U_i$, $i = 1,2$ be the responses of the market to the produced quantities of both players. Thus the realized quantities on the market for each of the players at moment t_1 will be

$$\begin{aligned}
p_1^{(1)} &= u_1^{(1)} - s_1^{(1)} = f_1\left(p_1^{(0)}, p_2^{(0)}, s_1^{(0)}\right) - Q_1\left(u_1^{(1)}, u_2^{(1)}\right)\\
&= f_1\left(p_1^{(0)}, p_2^{(0)}, s_1^{(0)}\right) - Q_1\left(f_1\left(p_1^{(0)}, p_2^{(0)}, s_1^{(0)}\right), f_2\left(p_1^{(0)}, p_2^{(0)}, s_2^{(0)}\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
p_2^{(1)} &= u_2^{(1)} - s_2^{(1)} = f_2(p_1^{(0)}, p_2^{(0)}, s_2^{(0)}) - Q_2(u_1^{(1)}, u_2^{(1)})\\
&= f_2(p_1^{(0)}, p_2^{(0)}, s_2^{(0)}) - Q_2(f_1(p_1^{(0)}, p_2^{(0)}, s_1^{(0)}), f_2(p_1^{(0)}, p_2^{(0)}, s_2^{(0)})).
\end{aligned}$$

We will define a new function, which we will call a generalized response function of the player and the market. Let $X \in X_1$, $Y \in X_2$, i.e. $X = (x, \delta x) \in P_1 \times s_1$ and $Y = (y, \delta y) \in P_2 \times s_2$. Let us denote $\overline{x} = (x, y, \delta x)$ and $\overline{y} = (x, y, \delta y)$.

$$F_1(X, Y) = F_1(x, y, \delta x, \delta y) = (f_1(\overline{x}) - Q_1(f_1(\overline{x}), f_2(\overline{y})), Q_1(f_1(\overline{x}), f_2(\overline{y})))$$

and

$$F_2(X, Y) = F_2(x, y, \delta x, \delta y) = (f_2(\overline{y}) - Q_2(f_1(\overline{x}), f_2(\overline{y})), Q_2(f_1(\overline{x}), f_2(\overline{y}))).$$

As far as in Assumption 3 the sets X and Y can be subsets of $\mathbb{R}^n$ we can reformulate Assumption 3 for the case of the generalized response function of the player and the market:

Assumption 4. *Let there is a duopoly market, satisfying the following assumptions:*

1. *the two firms are producing homogeneous goods that are perfect substitutes*
2. *the firm i, $i = 1,2$ can produce quantities from the set U_i, its set of the realized on the market production be P_i and the set of its surplus productions be s_i, where $X = P_1 \times s_1$ and $Y = P_2 \times s_2$ be closed, nonempty subsets of a complete metric space (Z, ρ)*

3. *let there exist a closed subset $D \subseteq X \times Y$ and maps $F_1 : D \to X$ and $F_2 : D \to Y$, such that $(F_1(x,y), F_2(x,y)) \subseteq D$ for every $(x,y) \in D$, be the generalized response function of the player and the market for firm one and two respectively*
4. *let there exists $\alpha \in (0,1)$, such that the inequality*

$$\rho(F_1(x,y), F_1(u,v)) + \rho(F_2(x,y), F_2(u,v)) \leq \alpha(\rho(x,u) + \rho(y,v)) \tag{17}$$

holds for all $(x,y), (u,v) \in X \times Y$.

Example 16. *Let $U_i = [0,+\infty)$, $P_i = [0,+\infty)$, $s_i = [0,+\infty)$, $X = P_1 \times s_1$ and $Y = P_2 \times s_2$. Let X and Y be subsets of $(\mathbb{R}^2, \rho)$, where*

$$\rho((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + |y_1 - y_2|.$$

Let $(X \times Y)$ be endowed with the metric $\widetilde{\rho}(\cdot,\cdot) = \rho(\cdot,\cdot) + \rho(\cdot,\cdot)$. Let $f_1 : X \times P_2 \to U_1$ and $f_2 : Y \times P_1 \to U_2$ be defined by

$$f_1(x,y,\delta x) = 45 - 0.5x + 0.25y - 0.1\delta x$$

and

$$f_2(x,y,\delta y) = 20 - 0.2x - 0.25y - 0.05\delta y,$$

where $(x,\delta x) \in X$ and $(y,\delta y) \in Y$.

Let the response functions of the market $Q_1 : U_1 \times U_2 \to U_1$ and $Q_2 : U_1 \times U_2 \to U_2$ be defined by

$$Q_1(x,y) = 0.05x + 0.03y \quad \text{and} \quad Q_2(x,y) = 0.04x + 0.06y.$$

Let the generalized response function of the player and the market $F_1 : X \times Y \to X$ and $F_2 : X \times Y \to Y$ be

$$F_1(x,y,\delta x,\delta y) = (f_1(\overline{x}) - Q_1(f_1(\overline{x}), f_2(\overline{y})), Q_1(f_1(\overline{x}), f_2(\overline{y})))$$

and

$$F_2(x,y,\delta x,\delta y) = (f_2(\overline{y}) - Q_2(f_1(\overline{x}), f_2(\overline{y})), Q_2(f_1(\overline{x}), f_2(\overline{y}))).$$

The equilibrium solution of the market are $x = 27.1$, $y = 9.6$, $\delta x = 1.6$ and $\delta y = 1.2$. The example show that in the equilibrium both players will have surplus productions greater that zero.

If we suppose that the players do not pay attention to the surplus quantities, i.e $F_1(x,y,\delta x) = 45 - 0.5x + 0.25y$ and $F_2(x,y,\delta y) = 20 - 0.2x - 0.25y$, we get an equilibrium solution in the market $x = 29.8$ and $y = 11.2$.

A variational technique in the investigation of equilibrium in duopoly markets

Definition 48. *Let $(Z, \preceq)$ be a partially ordered set, $X, Y \subseteq Z$ and $F : X \times Y \to X$, $f : X \times Y \to Y$ be semi-cyclic maps. The ordered pair (F, f) is said to have the mixed monotone property if*

$$\text{for any two } x_1, x_2, y \in X, \text{ such that } x_1 \preceq x_2 \text{ there holds the inequality} \\ F(x_1,y) \preceq F(x_2,y)$$

and

$$\text{for any two } y_1, y_2, x \in X, \text{ such that } y_1 \preceq y_2 \text{ there holds the inequality} \\ f(x,y_1) \succeq f(x,y_2).$$

A generalization of Ekeland's variational principle

We will use the notation $u = \big(u^{(1)}, u^{(2)}\big) \in Z \times Z$ and for any $u \in Z \times Z$ let us denote $\overline{u} = \big(u^{(2)}, u^{(1)}\big)$ just to fit some of the formulas in the text field.

Theorem 24. *Let $(Z, \rho, \preceq)$ be a partially ordered complete metric space, $(Z \times Z, d, \preceq)$, $X, Y \subseteq Z$ and $F : X \times Y \to X$ and $f : X \times Y \to Y$ be semi-cyclic, continuous maps with the mixed monotone property. Let*

$$V \times U = \{x = (x^{(1)}, x^{(2)}) \in X \times Y : x^{(1)} \preceq F(x) \text{ and } x^{(2)} \succeq f(x)\} \neq \emptyset.$$

Let $T : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be a proper, l.s.c, bounded from below function. Let $\varepsilon > 0$ be arbitrary and let $u_0 \in V \times U$ be an ordered pair so that the inequality $T(u_0) \leq \inf_{V \times U} T(v) + \varepsilon$ holds. Then there exists an ordered pair $x \in V \times U$, such that

(i) $T(x) \leq T(u_0)$

(ii) $d(x, u_0) \leq 1$

(iii) *for every $w \in V \times U$ different from $x \in V \times U$ holds the inequality $T(w) > T(x) - \varepsilon d(w, v)$.*

Coupled fixed points for semi–cyclic maps with the mixed monotone property

Theorem 25. *Let $(Z, \rho, \preceq)$ be a partially ordered complete metric space, $(Z \times Z, d, \preceq)$, $X, Y \subseteq Z$ and $F : X \times Y \to X$ and $f : X \times Y \to Y$ be an ordered pair of semi-cyclic maps with the mixed monotone property. Let there exists $\alpha \in [0, 1)$, so that the inequality*

$$\rho(F(x, y), F(u, v)) + \rho(f(x, y), f(u, v)) \leq \alpha\rho(x, u) + \alpha\rho(y, v)$$

holds for all $x \succeq u$ and $y \preceq v$. If there exists at least one ordered pair $(x, y) \in X \times Y$, such that $x \preceq F(x, y)$ and $y \succeq f(x, y)$, then there exists a coupled fixed points (x, y) of (F, f).

(25.i) *If in addition every pair of elements in $X \times Y$ has a lower or an upper bound then the coupled fixed point is unique.*

(25.ii) *if in addition every element in Z has a lower or an upper bound and $f(x, y) = F(y, x)$ then the coupled fixed point $(x, y))$ satisfies $x = y$.*

Applications ot Theorem 25 in equilibrium in duopoly partially ordered markets

Now we can restate Theorem 25 in terms of economic language.

Assumption 5. *Let us assume that two companies are offering products that are perfect substitutes. The first one can produce quantities from the set X and the second firm can produce quantities from the set Y, where X and Y be nonempty subsets of a partially ordered complete metric space $(Z, \rho, \preceq)$. Let $F : X \times Y \to X$, $f : X \times Y \to Y$ be the response functions of firm one and two, respectively. Let there exists $\alpha \in (0, 1)$, such that*

$$\rho(F(x, y), F(u, v)) + \rho(f(x, y), f(u, v)) \leq \alpha\rho(x, u) + \alpha\rho(y, v) \tag{18}$$

holds for all $x \succeq u$ and $y \preceq v$. If there exists at least one ordered pair $(x,y) \in X \times Y$, such that $x \preceq F(x,y)$ and $y \succeq f(x,y)$, then there exists a market equilibrium point (x,y), which is a coupled fixed points of (F,f).

If in addition every pair of elements in $X \times Y$ has an lower or an upper bound, then the coupled fixed point is unique.

The conditions imposed on the response functions states that we can say something only if when ever the production of firm one decreases i.e $x \succeq u$ the production of firm two increases i.e. $y \preceq v$. One case where it can happen is if in a monopoly market enters a second firm. In this case the first player will decrease its market share and the second one will increase it.

Example 17. *(Cournot's model) Let there be two companies producing a pair of products, which are perfect substitutes. Let us assume that the second player enters the market, so that outputs are (x_1,x_2) and (y_1,y_2). Then $(x_1,x_2) \succeq (y_1,y_2)$. Let endow the production set $\mathbb{R}$ with the euclidean norm $\|\cdot\|_2$. Consider the response functions $F(x_1,x_2,y_1,y_2)$ and $f(x_1,x_2,y_1,y_2)$ defined by*

$$F(x,y) = \begin{cases} \frac{x_1+y_1}{3} + 1 \\ \frac{x_2+y_2}{4} + 1 \end{cases}, \quad f(x,y) = \begin{cases} \frac{x_1+y_1}{3} + 1 \\ \frac{x_2+y_2}{2} + 1 \end{cases}.$$

Therefore there exists a market equilibrium, where production is $x_1 = 3$, $x_2 = 3$ for the first player and $y_1 = 3$, $y_2 = 5$ for the second one.

For response functions F and f if we try to apply the classical inequality for convex functions $\left(\frac{a+b}{2}\right)^2 \leq \frac{a^2+b^2}{2}$ then we will not be able to prove that inequality (18) holds true and we will not be able to proof that

$$\|F(X,Y) - F(U,V)\|_2 + \|f(X,Y) - f(U,V)\|_2 \leq \frac{5\sqrt{2}}{6}\|X-U\| + \frac{5\sqrt{2}}{6}\|Y-V\|.$$

holds for all X, Y, U, V and thereafter to aplly the results from the begining of the chapter.

Thus the consideration of a partially ordered metric space and that inequality (18) holds only for part of the elements of the space significantly increases the classes of oligopoly that can be investigated.

Example 18. *(Bertrand's model) Let us alter the Example 17, by assuming a market with two competing companies, each producing a single homogeneous product. The sole competitive advantage is the price. Let us assume that the second firm enters the market, i.e. if the productions are (x,p), x–quantity at a price of p and (y,q), y–quantity ate a price of q of the first and the second firm, respectively, then $(x,p) \succeq (y,q)$, assuming that the second firm is smaller, can produce at a higher prices. Let endow the production set $\mathbb{R}$ with the euclidean norm $\|\cdot\|_2$. Let us consider the response functions $F(x,p,y,q)$ and $f(x,p,y,q)$ defined in Example 1.*

Therefore there exists a market equilibrium. Actually the equilibrium production is $x = 3$ at a price $p = 3$ of the first player and $y = 3$ at a price $q = 5$ for the second player.

Coupled best proximity points for multi–valued maps

The results up to now about equilibrium in duopoly markets were obtained by assuming that each player will react to the market by choosing a singled value response. A more natural model will be obtained if we assume that each player may react by choosing some production from a set of possible reactions, i.e. if we consider the response functions to be a multi-valued functions.

Definition 49. *Let X and Y be two sets. The ordered pair (F_1, F_2) of multi–valued maps $F_1 : X \times Y \rightrightarrows X$ and $F_2 : X \times Y \rightrightarrows Y$ is called a semi–cyclic multi–valued map.*

Definition 50. *Let X and Y be two sets and (F_1, F_2) be a semi–cyclic multi–valued map. The ordered pair $(x^*, y^*) \in X \times Y$ is called a coupled fixed point for (F_1, F_2) if there holds $x^* \in F_1(x^*, y^*)$ and $y^* \in F_2(x^*, y^*)$.*

Theorem 26. *Let (X, ρ) and (Y, σ) be complete metric spaces, $F_1 : X \times Y \rightrightarrows X$, $F_2 : X \times Y \rightrightarrows Y$ and $\bar{x} \in X, \bar{y} \in Y$. Let there exists a constant $r > 0$ and $\alpha, \beta, \gamma, \delta \in [0, 1)$, satisfying $\max\{\alpha + \gamma, \beta + \delta\} < 1$ such that the following holds:*

(i) for all $(x, y) \in B_r(\bar{x}) \times B_r(\bar{y})$ the sets $F_1(x, y)$ and $F_2(x, y)$ are nonempty closed subsets of X and Y

(ii) $d(\bar{x}, F_1(\bar{x}, \bar{y})) + d(\bar{y}, F_2(\bar{x}, \bar{y})) < r(1 - \lambda)$, where $\lambda = \max\{\alpha + \gamma, \beta + \delta\}$

(iii) the inequality

$$\begin{aligned} S_2 &= e(F_1(x, y) \cap B_r(\bar{y}), F_1(u, v)) + e(F_2(z, w) \cap B_r(\bar{x}), F_2(t, s)) \\ &\leq \alpha\rho(x, u) + \beta\sigma(y, v) + \gamma p(z, t) + \delta\sigma(w, s), \end{aligned}$$

holds for all $(x, y), (u, v), (z, w), (t, s) \in B_r(\bar{x}) \times B_r(\bar{y})$.

Then, there exists at least ordered pair $(x, y) \in B_r(\bar{x}) \times B_r(\bar{y})$, which is a coupled fixed points for the semi–cyclic multi–valued map (F_1, F_2).

Example 19. *Let us choose $0 \leq \alpha < \beta < \gamma < \delta \leq \eta < +\infty$, $n, m \in (0, 1)$. Let us define the maps $f : [0, \delta] \to \left[\frac{\beta+\gamma}{2}, \gamma\right]$, $g : [\alpha, \eta] \to \left[\beta, \frac{\beta+\gamma}{2}\right]$, $\varphi : [0, \delta] \to \left[\frac{\beta+\gamma}{2}, \gamma\right]$, $\psi : [\alpha, \eta] \to \left[\beta, \frac{\beta+\gamma}{2}\right]$. by*

$$f(x) = \frac{\gamma - \beta}{2(\delta + 1)^n}(x + 1)^n + \frac{\beta + \gamma}{2},$$

$$g(x) = \frac{\gamma - \beta}{2((\eta + 1)^n - (\alpha + 1)^n)}(x + 1)^n + \beta - (\alpha + 1)^n \frac{\gamma - \beta}{2((\eta + 1)^n - (\alpha + 1)^n)},$$

$$\varphi(x) = \frac{\gamma - \beta}{2(\delta + 1)^m}(x + 1)^m + \frac{\beta + \gamma}{2},$$

$$\psi(x) = \frac{\gamma - \beta}{2((\eta + 1)^m - (\alpha + 1)^m)}(x + 1)^m + \beta - (\alpha + 1)^m \frac{\gamma - \beta}{2((\eta + 1)^m - (\alpha + 1)^m)}.$$

Let us denote $\bar{x} = \bar{y} = \frac{\beta+\gamma}{2}$, $\theta = \min\{|\delta - \bar{x}|, |\alpha - \bar{y}|\}$. Let us consider $\mathbb{R}$ with the canonical metric $|\cdot - \cdot|$. Let us denote the sets $X = [0, \delta]$, $Y = [0, \eta]$. lets us define the semi-cyclic multi-valued maps

$$F(x, y) = \{\xi : g(y) \leq \xi \leq f(x)\} \quad \text{and} \quad G(x, y) = \{\xi : \psi(y) \leq \xi \leq \varphi(x)\}.$$

A particular case can be obtained if $n = m = 1$, $\alpha = 0$, $\beta = 2$, $\gamma = 4$, $\delta = 6$ and $\eta = 8$. Then $f(x) = \varphi(x) = \frac{x}{7} + \frac{22}{7}$, $g(x) = \psi(x) = \frac{y}{8} + 2$, $r = 3$, $\overline{x} = \overline{y} = 3$ and

$$\begin{aligned} S_3 &= e(F(x,y) \cap B_r(\overline{x}), F(u,v)) + e(G(z,w) \cap B_r(\overline{y}), G(t,s)) \\ &\leq \tfrac{1}{8}|x-u| + \tfrac{1}{7}|y-v| + \tfrac{1}{8}|z-t| + \tfrac{1}{7}|w-s|. \end{aligned}$$

Example 20. *Let us consider the space $\mathbb{R}_p$. Let us choose $0 < \alpha_i < \beta_i < \gamma_i < \delta_i < \eta_i < +\infty$, $n_i, m_i \in (0,1]$ for $i = 1,2$, so that*

$$\max_{i=1,2}\left\{\frac{n_i(\gamma_i - \beta_i)}{2(\delta_i + 1)^{n_i}}\right\} + \max_{i=1,2}\left\{\frac{m_i(\gamma_i - \beta_i)}{2(\delta_i + 1)^{m_i}}\right\} < 1$$

and

$$\max_{i=1,2}\left\{\frac{n_i(\gamma_i - \beta_i)}{2((\eta_i + 1)^{n_i} - (\alpha_i + 1)^{n_i})}\right\} + \max_{i=1,2}\left\{\frac{m_i(\gamma_i - \beta_i)}{2((\eta_i + 1)^{m_i} - (\alpha_i + 1)^{m_i})}\right\} < 1.$$

Let us define the maps

$$f_i : [0, \delta_i] \to \left[\frac{\beta_i + \gamma_i}{2}, \gamma_i\right], \quad g_i : [\alpha_i, \eta_i] \to \left[\beta_i, \frac{\beta_i + \gamma_i}{2}\right],$$

$$\varphi_i : [0, \delta_i] \to \left[\frac{\beta_i + \gamma_i}{2}, \gamma_i\right], \quad \psi_i : [\alpha_i, \eta_i] \to \left[\beta_i, \frac{\beta_i + \gamma_i}{2}\right]$$

for $i = 1,2$ by

$$\begin{aligned} f_i(x) &= \frac{\gamma_i - \beta_i}{2(\delta_i + 1)^{n_i}}(x+1)^{n_i} + \frac{\beta_i + \gamma_i}{2}, \\ g_i(x) &= C(x+1)^{n_i} + \beta_i - (\alpha_i + 1)^{n_i} C, \\ \varphi_i(x) &= \frac{\gamma_i - \beta_i}{2(\delta_i + 1)^{m_i}}(x+1)^{m_i} + \frac{\beta_i + \gamma_i}{2}, \\ \psi_i(x) &= D(x+1)^{m_i} + \beta_i - (\alpha_i + 1)^{m_i} D, \end{aligned}$$

where $C = \frac{\gamma_i - \beta_i}{2((\eta_i+1)^{n_i} - (\alpha_i+1)^{n_i})}$ and $D = \frac{\gamma_i - \beta_i}{2((\eta_i+1)^{m_i} - (\alpha_i+1)^{m_i})}$.

Let us denote $\overline{x}_i = \frac{\beta_i + \gamma_i}{2}$ and $\theta_i = \min\{|\delta_i - \overline{x}_i|, |\alpha_i - \overline{x}_i|\}$ for $i = 1,2$. let us endow $\mathbb{R}^2$ with the metric $\rho((x,y),(u,v)) = \left(\left|\frac{x-u}{\theta_1}\right|^p + \left|\frac{y-v}{\theta_2}\right|^p\right)^{1/p}$, $p \in (1, +\infty)$. let us denote the sets $X_i = [0, \delta_i]$, $Y_i = [\alpha_i, \eta_i]$ for $i = 1,2$ and let $X = X_1 \times X_2$, $Y = Y_1 \times Y_2$. Let us define the semi-cyclic multi-valued maps $F : X \times Y \rightrightarrows X$ and $G : X \times Y \rightrightarrows Y$ by

$$F((x_1, x_2), (y_1, y_2)) = \{(\xi_1, \xi_2) : g_i(y_i) \leq \xi_i \leq f_i(x_i)\}$$

and

$$G((x_1, x_2), (y_1, y_2)) = \{(\xi_1, \xi_2) : \psi_i(y_i) \leq \xi_i \leq \varphi_i(x_i)\}.$$

A disadvantage of the presented models in the previous paragraphs is that players choose a fixed production at a fixed price. Actually the response of each player is any quantity form a set of possible productions or a price from possible prices. Therefore we will consider the response functions $F : X \times Y \rightrightarrows U \subset X$ and $f : X \times Y \rightrightarrows V \subset Y$ be multi–valued maps.

Assumption 6. *Let us assume that two companies are offering products that are perfect substitutes. The first one can produce quantities from the set X and the second firm can produce quantities from the set Y, where X and Y be nonempty subsets of a partially ordered complete metric space (Z,ρ) and $\bar{x} \in X, \bar{y} \in Y$. Consider $F : X \times Y \rightrightarrows X$ and $G : X \times Y \rightrightarrows Y$ to be the response function of players one and two, respectively. Let the ordered F and f satisfy the conditions of Theorem 26.*

Then there exists at least one market equilibrium point $(x,y) \in B_r(\bar{x}) \times B_r(\bar{y})$, which is a coupled fixed point for the ordered pair of response functions (F,G).

Example 21. *Let in Example 19 we consider two firms, producing one good, which is a perfect substitute. Let us put $\alpha = 10$, $\beta = 30$, $\gamma = 50$, $\delta = 80$ and $\eta = 100$ in Example 19. We may consider the interval $[0,\eta]$ as the set of the total production. Let the first firm be a smaller one and it production set is $[0,\delta]$ and the second one be a larger one with a production set $[\alpha,\eta]$. Let $n = 1$ and $m = 1/2$. Than for any initial start $[x,y]$ in the market the first firm chooses a production from the set $\left[\frac{y}{9} + \frac{260}{9}, \frac{10}{81}x + \frac{3250}{81}\right]$, the second firm from the set $\left[\frac{10}{9}\sqrt[2]{y+1} + 40, \frac{10\sqrt[2]{x+1}+30\sqrt[2]{101}-40\sqrt[2]{11}}{\sqrt[2]{101}-\sqrt[2]{11}}\right]$ and*

$$\begin{aligned} S_4 &= e(F(x,y) \cap B_r(\overline{x}), F(u,v)) + e(G(z,w) \cap B_r(\overline{y}), G(t,s)) \\ &\leq \frac{10}{81}|x-u| + \frac{1}{9}|y-v| + \frac{5}{9}|z-t| + \gamma|w-s|, \end{aligned}$$

where $\gamma = \frac{5}{\sqrt{101}-\sqrt{11}} < \frac{5}{6}$.

From $\max\left\{\frac{10}{81} + \frac{5}{9}, \frac{1}{9} + \frac{5}{6}\right\} = \max\left\{\frac{55}{81}, \frac{17}{18}\right\} = \frac{17}{18} < 1$ it follows that the pair of response functions satisfies Assumption 6 and consequently there exists an equilibrium pair of productions (x,y), such that $x \in F(x,y)$ and $y \in G(x,y)$.

Example 22. *Let us consider a model of a duopoly with two players, producing one good, which is a complete substitute, and let they compete on quantities and prices simultaneously. Let choose in Example 20, $\alpha_1 = 10$, $\beta_1 = 30$, $\gamma_1 = 40$, $\delta_1 = 60$, $\eta_1 = 100$, $\alpha_2 = 1$, $\beta_2 = 3$, $\gamma_2 = 4$, $\delta_2 = 5$, $\eta_2 = 8$ $n_1 = 1$, $n_2 = 1/2$, $m_1 = 1/2$, $m_2 = 1/4$. Let us consider the sets $X_i = [0,\delta_i]$, $Y_i = [\alpha_i,\eta_i]$ for $i = 1,2$ and let $X = X_1 \times X_2$, $Y = Y_1 \times Y_2$ and the multivalued maps $F : X \times Y \rightrightarrows X$ and $G : X \times Y \rightrightarrows Y$ from Example 20, which are the response functions of the two players, respectively, where the first coordinates are the response on the qualities and the second coordinate is the response on the price. Let us endow $\mathbb{R}^2$ with the metrics $\rho((x,y),(u,v)) = \left(\left|\frac{x-u}{\theta_1}\right|^p + \left|\frac{y-v}{\theta_2}\right|^p\right)^{1/p}$, $p \in (1,+\infty)$ from Example 20.*

There exists an equilibrium pair of productions and prices $((x,p),(y,q))$, such that $(x,p) \in F((x,p),(y,q))$ and $(y,q) \in G((x,p),(y,q))$.

Coupled best proximity points for semi–cyclic contraction pairs of maps

Definition 51. *Let A, B be nonempty subsets of a metric space (X,ρ) and (F,f), $F : A \times B \to A$, $f : A \times B \to B$ be a semi-cyclic ordered pair of maps. An ordered pair $(\xi,\eta) \in A \times B$ is called a coupled best proximity point of (F,f) if $\rho(\eta, F(\xi,\eta) = \rho(\xi, f(\xi,\eta) = \operatorname{dist}(A,B)$.*

Definition 52. *Let A, B be nonempty subsets of a metric space (X,ρ) and (F,f), $F : A\times B \to A$, $f : A\times B \to$ be a semi–cyclic ordered pair of maps. Let there exist a subset $D \subseteq A\times B$, such that $F : D \to A$, $f : D \to B$ and such that $(F(x,y), f(x,y)) \subseteq D$ for every $(x,y) \in D$. The semi–cyclic ordered pair of maps (F,f) is said to be a contraction of type two semi–cyclic ordered pair if there exist non-negative numbers α,β, such that $\alpha+\beta<1$ and there holds the inequality $\rho(F(x,y), f(u,v)) \le \alpha\rho(x,v)+\beta\rho(y,u)+(1-(\alpha+\beta))\mathrm{dist}(A,B)$ for all $(x,y),(u,v) \in D$.*

Definition 53. *Let A, B be nonempty subsets of the metric spaces (X,ρ), $F : A\times B \to A$, $f : A\times B \to B$ be semi–cyclic ordered pair of maps. For any pair $(x,y) \in A\times B$ we define the sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ by $x_0 = x$, $y_0 = y$ and $x_{n+1} = F(x_n,y_n)$, $y_{n+1} = f(x_n,y_n)$ for all $n \ge 0$.*

Coupled best proximity points for semi–cyclic contractions of type two

Simply to fit a few of the equations within the content field let us denote $d = \mathrm{dist}(A,B)$, $P_{n,m}(x,y) = \|x_n-y_m\|$ and $W_{n,m}(x,y) = P_{n,m}(x,y)-d = \|x_n-y_m\|-d$, where $x = \{x_n\}_{n=0}^{\infty}$ and $y = \{y_n\}_{n=0}^{\infty}$ are the iterated sequences defined in Definition 53.

Theorem 27. *Let A, B be nonempty subsets of a uniformly convex Banach space $(X,\|\cdot\|)$. Let there exist a subset $D \subseteq A\times B$ and semi–cyclic maps, so that $F : D \to A$ and $f : D \to B$, so that $(F(x,y), f(x,y)) \subseteq D$ for every $(x,y) \in D$. Let the ordered pair (F,f) be a semi–cyclic contraction of type two. Then (F,f) has a unique coupled best proximity point $(\xi,\eta) \in A\times B$, (i.e., $\|\eta-F(\xi,\eta)\| = \|\xi-f(\xi,\eta)\| = d$). For any initial guess $(x,y) \in A\times B$ there holds $\lim_{n\to\infty} x_n = \xi$, $\lim_{n\to\infty} y_n = \eta$, $\|\xi-\eta\| = d$, $\xi = F(\xi,\eta)$ and $\eta = f(\xi,\eta)$.*

If in addition $(X,\|\cdot\|)$ has a modulus of convexity of power type with constants $C>0$ and $q>1$, then

(i) A priori error estimates hold

$$\|\xi-x_m\| \le M_0\sqrt[q]{\frac{\max\{W_{0,1}(x,y),W_{0,0}(x,y)\}}{Cd}}\cdot\frac{\sqrt[q]{(\alpha+\beta)^m}}{1-\sqrt[q]{\alpha+\beta}};$$

$$\|\eta-y_m\| \le N_0\sqrt[q]{\frac{\max\{W_{0,1}(y,x),W_{0,0}(y,x)\}}{Cd}}\cdot\frac{\sqrt[q]{(\alpha+\beta)^m}}{1-\sqrt[q]{\alpha+\beta}};$$

(ii) A posteriori error estimates hold

$$\|\xi-x_n\| \le M_{n-1}\sqrt[q]{\frac{\max\{W_{n-1,n}(x,y),W_{n-1,n-1}(x,y)\}}{Cd}}\cdot\frac{\sqrt[q]{\alpha+\beta}}{1-\sqrt[q]{\alpha+\beta}};$$

$$\|\eta-y_n\| \le N_{n-1}\sqrt[q]{\frac{\max\{W_{n-1,n}(y,x),W_{n-1,n-1}(y,x)\}}{Cd}}\cdot\frac{\sqrt[q]{\alpha+\beta}}{1-\sqrt[q]{\alpha+\beta}},$$

where $M_n = \max\{\|x_n-y_n\|,\|x_n-y_{n+1}\|\}$, $N_n = \max\{\|x_n-y_n\|,\|y_n-x_{n+1}\|\}$.

Application of Theorem 27, when players' production sets have an empty intersection

When considering a duopoly, it is possible that the sets of possible productions of the two participants have an empty section. This possibility seems extreme at first glance, but it still exists. For example, one company has a huge production and holds a large market share and does not have the ability to maintain too low production levels. This is possible with long-term contracts.

Theorem 27 can be stated in the economic language.

Players' production sets have an empty intersection, each player is producing two goods

Example 23. *Let us consider a market with two competing firms, each firm produces two products and any one of the items is completely replaceable with a similar product of the other firm. Let us assume that the first firm can produce much less quantities than the second one, i.e., if x_1, x_2 are the quantities produced by the first firm and y_1, y_2 are the quantities produced by the second one and, then $x_1, x_2 \in [0,1]$ and $y_1, y_2 \in [2,3]$. Let $A = [0,1] \times [0,1]$ $B = [2,3] \times [2,3]$ be considered as subsets of $(\mathbb{R}^2, \|\cdot\|_2)$, which is a uniformly convex Banach space with modulus of convexity $\delta_{\|\cdot\|_2}(\varepsilon) \geq \frac{\varepsilon^2}{3}$ of power type. Let us consider the response functions $F(x_1, x_2, y_1, y_2)$ and $f(x_1, x_2, y_1, y_2)$ defined by*

$$F(x,y) = \begin{cases} \frac{3x_1}{8} + \frac{x_2}{8} - \frac{3y_1}{16} - \frac{y_2}{16} + 1 \\ \frac{x_1}{8} + \frac{3x_2}{8} - \frac{y_1}{16} - \frac{3y_2}{16} + 1 \end{cases}, \quad f(x,y) = \begin{cases} -\frac{3x_1}{16} - \frac{x_2}{16} + \frac{3y_1}{4} + \frac{y_2}{4} + \frac{5}{4} \\ -\frac{x_1}{16} - \frac{3x_2}{16} + \frac{y_1}{4} + \frac{3y_2}{4} + \frac{5}{4} \end{cases}.$$

There exists an equilibrium pair $(x,y) = ((x_1,x_2),(y_1,y_2))$ and for any initial start in the economy, the iterated sequence $(x^n, y^n) = ((x_1^n, x_2^n),(y_1^n, y_2^n))$ converges to the market equilibrium (x,y). We get in this case that the equilibrium pair of the production of the two firms is $x = (1,1)$, $y = (2,2)$ and the total production will be $a = (3,3)$.

Chapter V
TRIPLED FIXED POINTS AND TRIPLED BEST PROXIMITY POINTS

Examples are presented in [54] to show that it is possible to apply the coupled best proximity points for solving of non–symmetric systems of equations.

We will try to generalize the notion of tripled fixed points following the technique from [54]. The notion of tripled fixed points proposed in [9].

Notations and definitions

Let A_i, B_i for $i = 1,2,3$ be subset of X. Let us denote just to simplify the notations $A^3 = A_1 \times A_2 \times A_3$ and $B^3 = B_1 \times B_2 \times B_3$.

Definition 54. *Let A_i, B_i, $i = 1,2,3$ be six sets. We say that the ordered pair (F,G) of triples of maps $F = (F_1, F_2, F_3)$ and $G = (G_1, G_2, G_3)$ be a cyclic ordered pair of triple of maps if $F_i : A^3 \to B_i$, $G_i : B^3 \to A_i$ for $i = 1,2,3$.*

Just for the the sake of simplicity we will assume for the rest that the pair of maps (F, G) be a cyclic ordered pair of triple of maps, i.e. $(F, G) = ((F_1, F_2, F_3), (G_1, G_2, G_3))$.

Definition 55. *We say that $(\xi_1, \xi_2, \xi_3) \in A^3$ is a tripled fixed point for the ordered pair of triple of maps (F, G) if there holds $\xi_i = F_i(\xi_1, \xi_2, \xi_3)$, for $i = 1, 2, 3$.*

Let us denote $d_i = \operatorname{dist}(A_i, B_i)$ for $i = 1, 2, 3$, where A_i, B_i are subsets of a metric space (x, ρ).

Definition 56. *Let A_i, B_i, $i = 1, 2, 3$ be subsets of a metric space (X, ρ) and let (F, G) be a cyclic ordered pair of triple of maps. We say that $(\xi_1, \xi_2, \xi_3) \in A^3$ a tripled best proximity point of F if $\rho(\xi_i, F_i(\xi_1, \xi_2, \xi_3)) = d_i$, for $i = 1, 2, 3$.*

Definition 57. *Let A_i, B_i, $i = 1, 2, 3$ be six sets and let (F, G) be a cyclic ordered pair of triple of maps.*

For any triple $\left(x^{(1)}, x^{(2)}, x^{(3)}\right) \in A^3$ we define the sequences $\left\{x_n^{(i)}\right\}_{n=0}^{\infty}$, for $i = 1, 2, 3$ by $x_0^{(i)} = x^{(i)}$ for $i = 1, 2, 3$ and $x_{2n+1}^{(i)} = F_i\left(x_{2n}^{(1)}, x_{2n}^{(2)}, x_{2n}^{(3)}\right)$, $x_{2n+2}^{(i)} = G_i\left(x_{2n+1}^{(1)}, x_{2n+1}^{(2)}, x_{2n+1}^{(3)}\right)$ for $i = 1, 2, 3$ and for all $n \geq 0$.

If we consider $A_1 = A_2 = A_3 = B_1 = B_2 = B_3$ and $F_2(x, y, z) = F_1(y, z, x)$, $F_3(x, y, z) = F_1(z, x, y)$ and $G_i(x, y, z) = F_i(x, y, x)$ for $i = 1, 2, 3$ we get the sequence defined in [2].
If we consider $A_1 = A_2 = A_3 = B_1 = B_2 = B_3$ and $F_2(x, y, z) = F_1(y, x, y)$, $F_3(x, y, z) = F_1(z, y, x)$ and $G_i(x, y, z) = F_i(x, y, x)$ for $i = 1, 2, 3$ we get the sequence defined in [9].

Generalized cyclic contraction ordered pair of triple of maps

Just to simplify some of the statements let us agree that every where (F, G) be an ordered pair of ordered triples of maps, such that $F = (F_1, F_2, F_3)$, $F_i : A_1 \times A_2 \times A_3 \to B_i$ and $G = (G_1, G_2, G_3)$, $G_i : B_1 \times B_2 \times B_3 \to A_i$, for $i = 1, 2, 3$, where A_i, B_i for $i = 1, 2, 3$ be nonempty subsets of the underlying space X, metric or normed space.

Definition 58. *Let (X, ρ) be a metric space. We say that the cyclic ordered pair of triple of maps (F, G) be a generalized cyclic contraction of type one if there holds*

$$\sum_{i=1}^{3} \rho\left(F_i\left(x_i^{(1)}, x_i^{(2)}, x_i^{(3)}\right), G_i\left(y_i^{(1)}, y_i^{(2)}, y_i^{(3)}\right)\right) \leq \sum_{j=1}^{3} \sum_{i=1}^{3} \alpha_i^{(j)} \rho\left(x_i^{(j)}, y_i^{(j)}\right)$$

for some constants $\alpha_i^{(j)} \in [0, 1)$ for $i, j = 1, 2, 3$, so that $k = \max_{j=1,2,3}\left\{\sum_{i=1}^{3} \alpha_i^{(j)}\right\} < 1$ and for any $\left(x_i^{(1)}, x_i^{(2)}, x_i^{(3)}\right) \in A^3$, $\left(y_i^{(1)}, y_i^{(2)}, y_i^{(3)}\right) \in B^3$ for $i = 1, 2, 3$.

In what follows we will use the notation $d = \sum_{j=1}^{3} d_j = \sum_{j=1}^{3} \operatorname{dist}(A_j, B_j)$.

Definition 59. *Let (X, ρ) be a metric space. The ordered pair of triples of maps (F, G) is called generalized cyclic contraction pair of type two if the inequality*

$$\begin{array}{rcl} S_5 & = & \sum_{i=1}^{3} \rho\left(F_i\left(x_i^{(1)}, x_i^{(2)}, x_i^{(3)}\right), G_i\left(y_i^{(1)}, y_i^{(2)}, y_i^{(3)}\right)\right) \\ & \leq & \sum_{j=1}^{3} \sum_{i=1}^{3} \alpha_i^{(j)} \rho\left(x_i^{(j)}, y_i^{(j)}\right) + \sum_{j=1}^{3}\left(1 - \sum_{i=1}^{3} \alpha_i^{(j)}\right) d_j \end{array}$$

holds for some constants $\alpha_i^{(j)} \in [0,1)$, $i,j = 1,2,3$, *so that* $k = \max_{j=1,2,3}\left\{\sum_{i=1}^{3} \alpha_i^{(j)}\right\} < 1$ *and any* $\left(x_i^{(1)}, x_i^{(2)}, x_i^{(3)}\right) \in A^3$, $(y_i^{(1)}, y_i^{(2)}, y_i^{(3)}) \in B^3$ *for* $i = 1,2,3$.

Tripled fixed points

Theorem 28. *Let* A_i, B_i, $i = 1,2,3$ *be nonempty, closed subsets of a complete metric space* (X,ρ). *Let the ordered pair* (F,G) *be a generalized cyclic contraction of type* 1. *Then*

(i) There exists a unique triple (ξ_1,ξ_2,ξ_3) *in* $(A_1 \cap B_1) \times (A_2 \cap B_2) \times (A_3 \cap B_3)$, *which is a common triple fixed point for the maps* F *and* G. *Moreover the iteration sequences* $\left\{x_i^{(n)}\right\}_{n=0}^{\infty}$, *for* $i = 1,2,3$, *defined in Definition 57 converge to* ξ_i *for* $i = 1,2,3$ *respectively for any initial guess triple* $\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right)$;

(ii) a priori error estimates hold $\max\left\{\rho(x_i^{(n)},\xi_i) : i = 1,2,3\right\} \le \frac{k^n}{1-k}\sum_{i=1}^{3}\rho\left(x_i^{(1)}, x_i^{(0)}\right)$

(iii) a posteriori error estimates hold $\max\left\{\rho(x_i^{(n)},\xi) : i = 1,2,3\right\} \le \frac{k}{1-k}\sum_{i=1}^{3}\rho\left(x_i^{(n-1)}, x_i^{(n)}\right)$

(iv) the rate of convergence is given by $\sum_{i=1}^{3}\rho\left(x_i^{(n)},\xi_i\right) \le k\sum_{i=1}^{3}\rho\left(x_i^{(n-1)},\xi_i\right)$.

Example 24. *Let us consider the system of nonlinear equations:*

$$\left|\begin{array}{rcccccc} -9x & + & e^{y-1} & + & 3\mathrm{arctg}(z-2) & = & 0 \\ -24x+3x^2 & + & e^{(y-1)^2} & + & 3\mathrm{arctg}((z-2)^2) & = & -36 \\ -36x+3x^3 & + & e^{(y-1)^3} & + & 3\mathrm{arctg}((z-2)^3) & = & -90. \end{array}\right. \tag{19}$$

We consider the functions $F_1(x,y,z) = \frac{x}{4} + \frac{e^{y-1}}{12} + \frac{\mathrm{arctg}(z-2)}{4}$, $F_2(x,y,z) = \frac{x^2}{8} + \frac{e^{(y-1)^2}}{24} + \frac{\mathrm{arctg}((z-2)^2)}{8} + 1.5$, $F_3(x,y,z) = \frac{x^3}{12} + \frac{e^{(y-1)^3}}{36} + \frac{\mathrm{arctg}((z-2)^3)}{12} + 2.5$, the sets $A_1 \times A_2 \times A_3 = [0,1]\times[1,2]\times[2,3]$ and we apply Theorem 28.

Table 15. Number m of iterations needed by the a priori estimate $(0,1,2)$

ε	0.1	0.01	0.001	0.0001	0.00001	0.000001
m	14	22	30	38	56	54

Table 16. Number m of iterations needed by the a posteriori estimate $(0,1,2)$

ε	0.1	0.01	0.001	0.0001	0.00001	0.000001
m	5	7	10	12	14	16

We get an approximation of the solution $(0.3741116328, 1.615504553, 2.553428358)$.

Tripled best proximity points

Theorem 29. *Let A_i, B_i, for $i = 1,2,3$ be nonempty, closed and convex subsets of a uniformly convex Banach space. Let the ordered pair (F,G) of ordered triples of maps be a generalized cyclic contraction of type 2. Then F has a unique tripled best proximity point $(\xi_1,\xi_2,\xi_3) \in A^3$ and G has a unique tripled best proximity point $(\upsilon_1,\upsilon_2,\upsilon_3) \in B^3$, i.e. $\|\xi_i - F_i(\xi_1,\xi_2,\xi_3)\| = d_i$, $\|\upsilon_i - G_i(\upsilon_1,\upsilon_2,\upsilon_3)\| = d_i$, $i = 1,2,3$) and*

$$\begin{gathered} G_i(F_1(\xi_1,\xi_2,\xi_3),F_2(\xi_1,\xi_2,\xi_3),F_3(\xi_1,\xi_2,\xi_3)) = \xi_i, \\ F_i(G_1(\upsilon_1,\upsilon_2,\upsilon_3),G_2(\upsilon_1,\upsilon_2,\upsilon_3),G_3(\upsilon_1,\upsilon_2,\upsilon_3)) = \upsilon_i \end{gathered} \tag{20}$$

for $i = 1,2,3$.

Moreover $\upsilon_i = F_i(\xi_1,\xi_2,\xi_3)$ and $\xi_i = G_i(\upsilon_1,\upsilon_2,\upsilon_3)$ for $i = 1,2,3$. For any arbitrary point $\left(x_1^{(0)},x_2^{(0)},x_2^{(0)}\right)$ there hold $\lim_{n\to\infty} x_i^{(2n)} = \xi_i$, $\lim_{n\to\infty} x_i^{(2n+1)} = \upsilon_i$ for $i = 1,2,3$ and $\sum_{i=1}^{3} \|\xi_i - \upsilon_i\| = d$.

If in addition the modulus of convexity δ is of power type with constants $C > 0$ and $q > 1$ then tehre holds the error estimates:

(i) a priori error estimates hold $\max\left\{\left\|\xi_i - x_i^{(2m)}\right\| : i = 1,2,3\right\} \le P_{0,1}(x)\sqrt[q]{\frac{W_{0,1}(x)}{Cd}} \cdot \frac{\left(\sqrt[q]{k^2}\right)^m}{1-\sqrt[q]{k^2}}$

(ii) a posteriori error estimates hold

$$\max\left\{\left\|\xi_i - x_i^{(2n)}\right\| : i = 1,2,3\right\} \le P_{2n,2n-1}(x)\sqrt[q]{\frac{W_{2n,2n-1}(x)}{Cd}} \cdot \frac{\sqrt[q]{k}}{1-\sqrt[q]{k^2}},$$

where $x = (x_1,x_2,x_3)$ and $x_i = \{x_i^{(n)}\}_{n=0}^{\infty}$ for $i = 1,2,3$ are the sequences defined in Definition 57.

Variant of results from [2, 9, 47] are corollaries of Theorem 29.

A market equilibrium in oligopoly with three player

We will present a generalization of the duopoly model by looking at an oligopoly with three companies.

Following [24], let us assume that three companies competing for one and the same consumer segment and striving to meet the demand with overall production Market players follow cost functions $c_i(x_i)$, $i = 1,2,3$, respectively. Assuming that three firms are acting rationally, the payoff functions are $\Pi_i(x) = x_i P(\sum_{i=1}^{3} x_i) - c_i(x_i)$, for $i = 1,2,3$ of the three firms, respectively.

The goal of each company is to maximize its payoff, i.e.

$$\max\{\Pi_i(x) : x_i, \text{ assuming that } x_j \text{ for } j \ne i \text{ are fixed}\}.$$

Provided that functions P and c_i, $i = 1,2,3$ are differentiable, we get the system of equations

$$\frac{\partial \Pi_i(x)}{\partial x_i} = P\left(\sum_{i=1}^{3} x_i\right) + x_i P'\left(\sum_{i=1}^{3} x_i\right) - c_i'(x_i) = 0, \quad i = 1,2,3 \tag{21}$$

The market equlibrium is a solution of (21) [24].

We can find an implicit formula for the response function in (21) $x_i = \frac{c_i'(x_i)-P(\sum_{i=1}^3 x_i)}{P'(\sum_{i=1}^3 x_i)} = F_i(x)$ for $i=1,2,3$ and to transform the problem of maximization of the payoffs to a problem of trepled fixed points.

Semi–cyclic ordered triple of maps

Definition 60. *Let A_i, $i=1,2,3$ be nonempty subsets of a metric space (X,ρ) and $F=(F_1,F_2,F_3)$ be an ordered triple of maps. If $F_i : A_1\times A_2\times A_3 \to A_i$, $i=1,2,3$, the the ordered triple of maps $F=(F_1,F_2,F_3)$ is called a semi–cyclic maps.*

Definition 61. *Let A_i, $i=1,2,3$ be nonempty subsets of a metric space (X,ρ) and (F_1,F_2,F_3) be a semi–cyclic ordered triple of maps, i.e. $F_i : A_1\times A_2\times A_3 \to A_i$, $i=1,2,3$. An ordered triple $x=(x_1,x_2,x_3)\in A_1\times A_2\times A_3$ is called a tripled fixed point of (F_1,F_2,F_3) if $x_i = F_i(x)$ for $i=1,2,3$.*

Definition 62. *Let A_i, $i=1,2,3$ be nonempty subsets of X. Let $F_i : A_1\times A_2\times A_3 \to A_i$,$i=1,2,3$. For any pair $\left(x_1^{(0)},x_2^{(0)},x_3^{(0)}\right)\in A_1\times A_2\times A_3$ we define the sequences $\left\{x_i^{(n)}\right\}_{n=0}^{\infty}$, $i=1,2,3$ by $x_i^{(n+1)} = F_i\left(x_i^{(n)},x_i^{(n)},x_i^{(n)}\right)$ for all $n\ge 0$. and all $i=1,2,3$.*

Definition 63. *Let A_i, $i=1,2,3$ be nonempty subsets of a metric space (X,ρ). Let there exist a subset $D\subseteq A_1\times A_2\times A_3$ and maps $F_i : D\to A_i$ for $i=1,2,3$, such that $(F_1(x),F_2(x),F_3(x))\subseteq D$ for every $x\in D$. The ordered pair of ordered triple (F_1,F_2,F_3) is said to be a cyclic contraction ordered tripled if there exist non-negative numbers $\alpha_i,\beta_i,\gamma_i$, $i=1,2,3$, such that $\max\{\sum_{i=1}^3\alpha_i,\sum_{i=1}^3\beta_i,\sum_{i=1}^3\gamma_i\}<1$ and there holds the inequality*

$$\sum_{i=1}^{3}\rho(F_i(x_i,y_i,z_i),F_i(u_i,v_i,t_i)) \le \sum_{i=1}^{3}\left(\alpha_i\rho(x_i,u_i)+\beta_i\rho(y_i,v_i)+\gamma_i\rho(z_i,t_i)\right)$$

for all $(x_i,y_i,z_i),(u_i,v_i,t_i)\in D$, $i=1,2,3$.

Theorem 30. *Let A_i, $i=1,2,3$ be nonempty and closed subsets of a complete metric space (X,ρ). Let there exist a closed subset $D\subseteq A_1\times A_2\times A_3$ and maps $F_i : D\to A_i$, $i=1,2,3$, such that $(F_1(x,y,z),F_2(x,y,z),F_3(x,y,z))\subseteq D$ for every $(x,y,z)\in D$. Let the ordered tripled (F_1,F_2,F_3) be a cyclic contraction ordered tripled. Then*

(i) there exists a unique pair (ξ_1,ξ_2,ξ_3) in D, which is a unique tripled fixed point for the ordered pair (F_1,F_2,F_3) and the iteration sequences $\left\{x_i^{(n)}\right\}_{n=0}^{\infty}$, $i=1,2,3$, defined in Definition 62 converge to ξ_i, respectively, for any arbitrary chosen initial guess $\left(x_1^{(0)},x_2^{(0)},x_3^{(0)}\right)\in D$

(ii) a priori error estimates hold $\max\{\rho(x_i^{(n)},\xi_i),i=1,2,3\}\le\frac{k^n}{1-k}\sum_{i=1}^3\rho(x_i^{(1)},x_i^{(0)})$

(iii) a posteriori error estimates hold $\max\{\rho(x_i^{(n)},\xi_i),i=1,2,3\}\le\frac{k}{1-k}\sum_{i=1}^3\rho(x_i^{(n-1)},x_i^{(n)})$

(iv) the rate of convergence for the sequences of successive iterations

$$\sum_{i=1}^{3} \rho(x_i^{(n)}, \xi_i) \leq k \sum_{i=1}^{3} \rho(x_i^{(n-1)}, \xi_i),$$

where $k = \max\{\sum_{i=1}^{3} \alpha_i, \sum_{i=1}^{3} \beta_i, \sum_{i=1}^{3} \gamma_i\}$.

If in addition $F_2(x, y, z) = F_1(y, z, x)$ *and* $F_3(x, y, z) = F_1(z, x, y)$ *(i.e. the players have a symmetric type of response functions) then the tripled fixed point* (x, y, z) *satisfies* $x = y = z$.

Existence and uniqueness of market equilibrium in an oligopoly with three players

Example 25. *Let us first start with an oligopoly model with three industrial companies competing for one and the same consumer segment and striving to meet the demand with the overall industrial production.*

Let the response functions of each of the players be

$$F_1(x_1, x_2, x_3) = \frac{90}{4} - \frac{x_2 + x_3}{4}, \quad F_2(x_1, x_2, x_3) = \frac{80}{6} - \frac{x_1 + x_3}{6}, \quad F_3(x_1, x_2, x_3) = \frac{70}{8} - \frac{x_2 + x_3}{8}$$

and $A_1 = [0, 30]$, $A_2 = [0, 40]$, $A_3 = [0, 50]$.

There exists an equilibrium pair (ξ_1, ξ_2, ξ_3) and for any initial start in the economy the iterated sequences $(x_1^{(0)}, x_2^{(0)}, x_3^{(0)})$ converge to the market equilibrium (ξ_1, ξ_2, ξ_3). The equilibrium triple is $\left(\frac{415}{22}, \frac{205}{22}, \frac{115}{22}\right)$.

Conclusion

Summary of the obtained results

The main contributions in the present thesis

I. Ekeland's variational principle for maps with the mixed monotone property is generalized. With the help of it, conditions for the existence and conditions for the uniqueness of coupled fixed points for classes of maps with the mixed monotone property are found. The classes of maps with the mixed monotone property, for which coupled fixed points exist, have been extended using the generalized variation principle.

II. An error estimation technique for best proximity points, developed in [52] is presented, using sequences of successive iterations. This technique was used to find the error estimate for coupled and tripled best proximity points.

III. It is proven, that for the considered up to now cyclic maps their coupled fixed point and coupled best proximity points (x, y) have to satisfy $x = y$. A generalization of the notions of ordered pairs of cyclic maps, coupled fixed points and coupled best proximity points are introduced and are named modified cyclic map, modified coupled fixed points and modified coupled best proximity points, respectively. This new class of ordered pairs of cyclic maps can be used in solving of non-symmetric systems of transcendental equations, for which the known numerical methods, used in algebraic computer system Maple 18.00 coupled not present the exact solution.

IV. The notion of coupled best proximity point is generalized for maps in modular function spaces. Using the possible generalizations of the modulus of convexity in modular function spaces generalizations of key lemmas of Eldred and Veermani are proven. The technique for the investigation of best proximity points in modular function spaces, developed in [51], is applied in the investigation of coupled best proximity points in modular function spaces. An Illustration is presented for an application in solving of systems of transcendental equations, for which the algebraic computer system Maple 18.00 could not find the exact solution.

V. The notion of semi-cyclic maps has been introduced, which naturally arises when studying market equilibrium in oligopoly markets. A new model for studying the existence and uniqueness of market equilibrium in duopoly markets, which is based on the response functions, is presented. The illustrations demonstrate its advantages over the classical model for maximizing payoff functions by eliminating the need for differentiation, studying the contour of the set of possible productions and obtaining conditions for stability of the sequence of successive productions.

VI. The possibility of generalizing some of the studied problems in Chapters 1 to 4 for tripled fixed points and tripled best proximity points in the study of oligopoly markets with three players, using semi-cyclic maps of three variables is considered.

The main contributions in the present thesis are:

List of publications included in the thesis

1 L. Ajeti, B. Zlatanov: Coupled fixed points results for Hardy–Rogers type of maps with the mixed monotone property obtained with the help of a variational technique. *MATTEX 2022, CONFERENCE PROCEEDING*, ISSN: 1314-3921, (2022) (to appear).

2 Y. Dzhabarova, S. Kabaivanov, M. Ruseva, B. Zlatanov: Existence, Uniqueness and Stability of Market Equilibrium in Oligopoly Markets, *Administrative Sciences* **10**(3), Article number 70, (2020), ISSN:2076-3387 (Web of Science, SCOPUS)

3 Y. Dzhabarova, B. Zlatanov: A Note on the Market Equilibrium in Oligopoly with Three Industrial Players, *AIP Conference Proceedings*, (Web of Science, SCOPUS, SJR=0.19) (to appear),

4 G. Gecheva, M. Hristov, D. Nedelcheva, M. Ruseva, B. Zlatanov, Applications of Coupled Fixed Points for Multivalued Maps in the Equilibrium in Duopoly Markets and in Aquatic Ecosystems. *Axioms* **10**(2), Article number 44, (2021), ISSN 2075-1680 (Web of Science IF=1.824, Q2, SCOPUS, SJR=0.314, SCOPUS, SJR=0.314, Q3)

5 M. Hristov, A. Ilchev, B. Zlatanov: Coupled fixed points for Chatterjea type maps with the mixed monotone property in partially ordered metric spaces. *AIP Conference Proceedings*, 2172, Article number 060003 (2019), ISSN 0094-243X, ISSN 1551-7616, (Web of Science, SCOPUS, SJR=0.182)

6 M. Hristov, A. Ilchev, B. Zlatanov: On some application on coupled and best proximity points theorems. *AIP Conference Proceedings*, 2333, Article number 080008 (2021), ISSN 0094-243X, ISSN 1551-7616 (Web of Science, SCOPUS, SJR=0.19)

7 M. Hristov, A. Ilchev, D. Nedelcheva, B. Zlatanov: Existence of Coupled Best Proximity Points of p–Cyclic Contractions. *Axioms*, **10**(1), Article number 39, (2021), ISSN 2075-1680, (Web of Science IF=1.824, Q2, SCOPUS, SJR=0.314, Q3)

8 A. Ilchev, B. Zlatanov: Coupled Fixed Points and Coupled best Proximity Points in Modular Function Spaces, *International Journal of Pure and Applied Mathematics* **118**(4), (2018) 957-977, ISSN: 1311-8080 (printed), ISSN: 1314-3395 (online)

9 A. Ilchev, B. Zlatanov: Coupled Fixed Points and Coupled Best Proximity Points for Cyclic Kannan Type Contraction Maps in Modular Function Spaces. *MATTEX 2018, CONFERENCE PROCEEDING*, **v.1**, 75–88, (2018), ISSN: 1314-3921

10 A. Ilchev, B. Zlatanov: Error estimates for approximation of coupled best proximity points for cyclic contractive maps, Applied Mathematics and Computation, **290**, 412–425 (2016), ISSN: 0096-3003 (printed), ISSN:1873-5649 (online); (Web of Science, IF=1.738, Q1; SCOPUS, SJR=0.944, Q1, Zbl 1410.41010, MR3523439)

11 S. Kabaivanov, V. Zhelinski, B. Zlatanov. Coupled Fixed Points for Hardy–Rogers Type of Maps and Their Applications in the Investigations of Market Equilibrium in Duopoly Markets for Non-Differentiable, Nonlinear Response Functions, *Symmetry* **14**(3), Article number 605, (2022), ISSN: 2073-8994 (online), (Web of Science, IF=2.94, Q2; SCOPUS, SJR=0.385, Q2)

12 S. Kabaivanov, B. Zlatanov: A variational principle, coupled fixed points and market equilibrium, *Nonlinear Analysis: Modelling and Control*, **26**(1), 169-185, , (2021), ISSN: 1392-5113 (printed), ISSN: 2335-8963 (online); (Web of Science, IF=2.217, Q1; SCOPUS, SJR=0.757, Q2, Zbl 07357592, MR4195512).

13 B. Zlatanov: Best proximity points in modular function spaces, *Arabian Journal of Mathematics*, **4**(3), 215–227 (2015), ISSN: 2193-5343 (printed), ISSN: 2193-5351 (online); (Web of Science, SCOPUS, Zbl 1325.47103, MR3399285)

14 B. Zlatanov: Error Estimates for Approximating of Best Proximity Points for Cyclic Contractive Maps, Carpathian J. Math., **32**(2), 241–246 (2016), Carpathian Journal of mathematics **32**(2), 265-270, (2016), ISSN:1843-4401 (online); (Web of Science, IF=0.788, Q2; SCOPUS, SJR=0.457, Q2, Zbl 1424.46023, MR3587895)

15 B. Zlatanov: A Variational Principle and Coupled Fixed Points, *Journal of Fixed Point Theory and Applications* 21:69, (2019), ISSN:1661-7738, (printed) ISSN:1661-7746 (online); (Web of Science, IF=1.741, Q1; SCOPUS, SJR=0.642, Q2, Zbl 1474.54291, MR3950777)

16 B. Zlatanov: Coupled best proximity points for cyclic contractive maps and their applications, Fixed Point Theory, **22**(1), 431-452 (2021), ISSN 1583-5022, ISSN (online) 2066-9208 (online); (Web of Science, IF=1.396, Q1; SCOPUS, SJR=0.68, Q2, MR4269039, Zbl 07370686).

17 B. Zlatanov: On a Generalization of Tripled Fixed or Best Proximity Points for a Class of Cyclic Contractive Maps, FILOMAT, **35**(9), 3015-3031, (2021), ISSN 0354-5180, (Web of Science, IF=0.988, Q2; SCOPUS, SJR=0.449, Q2, MR4365419)

18 B. Zlatanov: On some applications of coupled fixed (or best proximity) points, *MATTEX 2020, CONFERENCE PROCEEDING, 25–27 October 2020*, **v.1**, 3–19, (2020), ISSN: 1314-3921

Approbation of the obtained results

Some of the results were presented in an invited talk at *MATTEX 2020, CONFERENCE PROCEEDING, 22–24 October 2020* [53].

a) L. Ajeti, B. Zlatanov: Coupled fixed points results for Hardy–Rogers type of maps with the mixed monotone property obtained with the help of a variational technique, *MATTEX 2022, Shumen, Bulgaria, 12–14 May 2022*

b) Y. Dzhabarova, B. Zlatanov: A Note on the Market Equilibrium in Oligopoly with Three Industrial Players, *Tenth International Conference Techsys 2021, Plovdiv, Bulgaria 27–29 May 2021*

c) M. Hristov, A. Ilchev, B. Zlatanov: Coupled fixed points for Chatterjea type maps with the mixed monotone property in partially ordered metric spaces. 45*th International Conference on Applications of Mathematics in Engineering and Economics, AMEE 2020, Sozopol, Sofia, 7-13 June 2019*

d) M. Hristov, A. Ilchev, B. Zlatanov: On some application on coupled and best proximity points theorems. 46*th International Conference on Applications of Mathematics in Engineering and Economics, AMEE 2020, Sozopol, Sofia, 7-13 June 2020*

e) A. Ilchev, B. Zlatanov: Coupled Fixed Points and Coupled Best Proximity Points for Cyclic Kannan Type Contraction Maps in Modular Function Spaces, *MATTEX 2018, Shumen, Bulgaria, 22–24 October 2018*

f) B. Zlatanov: On some applications of coupled fixed (or best proximity) points, *MATTEX 2020, Shumen, Bulgaria, 22–24 October 2020*

The connection between the contributions, the tasks, the paragraphs in the thesis and the included publications.

The connection between the contributions, the tasks, the paragraphs in the thesis and the included publications is as follows:

Contribution	Chapter	Paragraph	Publication	Conference report
I	1	4.5	2,3,5,6	a),c),f)
II	1,5	4.2,4.3,4.4,4.7	2,6,7,10,15,17,18	d),f)
III		2.3	17	d),f)
IV	3		5,8,9,11,14	e),f)
V	4,5		2,3,4,12,13	f)
VI	5		2,18	b)

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