R E V I E W

by DSc Angel Borisov Dishliev
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on

the Thesis for awarding the scientific degree "Doctor of Sciences";
Field of Higher Education: 4. Natural sciences, Mathematics, and Informatics;
Professional Field: 4.5. Mathematics (Mathematical Analysis);
Author of the Thesis: Prof. Dr. Boyan Georgiev Zlatanov;
Title of the Thesis: "Applications of Coupled Fixed Points and Coupled Best Proximity Points"

By order No. RD-21-1333 of 18.07. 2022 of the Rector of University "Paisii Hilendarski" of Plovdiv (PU), I was appointed as a member of the Scientific Jury under the procedure for awarding the above-mentioned dissertation work. At the first meeting of the jury, I was selected as a reviewer.

In this review, I will follow the guidelines written in the Rules for the Development of the Academic Staff (RDAS) of PU.

1. General description of the presented materials

The set of electronic media materials presented by Prof. Boyan Zlatanov is in accordance with Art. 36 (1) of the RDAS of PU. The set contains a significant number of documents, the most important of which are:

- Dissertation work;
- Abstract;
- Declaration of originality and authenticity;
- CV (European format);
- Copies of 18 scientific articles presented in full text. Scientific articles are related to the topic of the dissertation work. The candidate for obtaining the scientific degree "Doctor of Sciences" is a member of the authors' collectives of these articles. The Thesis are based on these publications;
- Certificate of fulfillment of the minimum national requirements;
- List of citations;
- Other documents that are related to the procedure for the defense of the dissertation work and are required by the relevant regulations**.**

Of course, the main paper is the thesis entitled "Applications of Coupled Fixed Points and Coupled Best Proximity Points", placed on 313 standard pages. It is consists of:

- Preface;
- Introduction;
- Five chapters (each containing several paragraphs);
- Conclusion containing:
  -- Main contributions in the dissertation work – 6 main contributions are modestly indicated,
  -- List of publications on the dissertation work – 18 publications in total,

-- Declaration of originality of the results;

-- Approbation of the results - five participations in international scientific conferences with reports that were published in the relevant scientific volumes of the conferences (I report some conferences where the results are reported: TechSys 2021 - Engineering, Technologies and Systems; International Conference on Applications of Mathematics in Engineering and Economic; MATTECH),

- Bibliography including 132 titles.

2. Brief biographical data

The candidate for acquiring the scientific degree "Doctor of Sciences" completes the following educational qualification degrees consecutively over time:

Period	Degree	Qualification and Specialization	School
1986-1991	Secondary		Language high school “Georgi Kirkov”, Plovdiv
1991-1996	Bachelor and master de-grees	Mathematics Mathematical Analyses; Teacher in Math and Informatics	SU "St. Kliment Ohridski", FMI
1997-2001	PhD Thesis	Mathematics 4.5 (mathematical analysis doctoral program)	PU "Paisii Hilendarski", FMI

Table 1

The title of PhD Thesis of Dr. B. Zlatanov for obtaining the educational and scientific degree "Doctor" is: "Geometric Properties of Some Classes of Banach Spaces with an Unconditional Basis", Education Area: 4. Natural Sciences, Mathematics and Informatics, Professional Direction 4.5. Mathematics, Doctoral Program "Mathematical Analysis". The dissertation was defended in 2001.

The professional realization of the candidate is in PU. The positions held, successively over time, are indicated in the following table:

Period	Position	University
1999 – 2001	Instructor in Mathematical Analysis	PU "Paisii Hilendarski"
2001 – 2008	"Assistant'' "Senior Assistant", "Chief Assistant"	PU "Paisii Hilendarski"
2008 – 2019	„Associate professor“	PU "Paisii Hilendarski"
2019 - …	„Professor“	PU "Paisii Hilendarski"
2015 - …	Vice Dean FMI	PU "Paisii Hilendarski"“

Table 2

The academic positions "Associate Professor" and "Professor" are in the Higher Education Field: 4. Natural Sciences, Mathematics and Informatics, and Professional Field: 4.5. Mathematics (Mathematical Analysis).

The level and merits of research activity of one scientict are most accurately and objectively determined by his achievements. Boyan Zlatanov has 67 scientific articles. Some of these articles are submitted for the acquisition of the degrees "Doctor" and "Doctor of Sciences", for the occupation of the academic positions "Associate professor" and "Professor", and some of them are not used. Their distribution (as numbers) is given in more detail in the following table:

Scientific publications presented for participation in a procedure for	Number of publications
PhD thesis	4
Associate Professor	5
Professor	36
Doctor of Science	16
Articles which are not used in the procedures	6

Table 3

- 25 scientific articles are published in the journals with impact factor (IF). The publications in journals with Impact rank (SJR) are 33. The candidate's creativity can be conditionally distributed in several scientific directions:
- Geometric properties in Banach spaces;
- Fixed points of operators in Banach spaces;
- Mathematical models and application of mathematics;
- Mathematics training.

One of the most important qualities of the scientific papers is the effect that his work causes in the scientific community. Quantitatively, this is most clearly expressed through the number of citations. They are qualitatively classified by their level, more precisely, through the journal level where the publications are cited.

A total of 248 citations are indicated in the relevant reference (List of all noticed citations) accompanied by a complete and accurate description. The object of citation are 53 scientific works of Prof. B. Zlatanov. All the citations are given according to accepted standards, i.e. self-citations are excluded. I will note that:

- 78 citations are in the scientific articles from the journals with IF. Only three of these 78 citations are with the co authors Bulgarian scientists;
- 115 citations are in the articles published in journals with SJR.

Citation 102 in the List of the citations observed (this list is found in the author's submissions for the competition), the rank of the journal Mathematics, which is SJR=0.244 (in the year of publication) is not noted. I have to clarify that according to the Regulations for Implementation of Act on Development of the Academic Staff in the Republic of Bulgaria (ADASRB), one publication has SJR if the journal (in which is published) is indexed for the relevant year in SCOPUS and Web of Science. The journal mentioned above is not indexed in WoS. However, I believe that the publication can be counted with the corresponding rank. Also, in the year of publication, the journal is indexed Q3, not Q2, as noted by the author.

It is clear that the number of citations is a monotonically increasing, it is a stepwise integer function of time, in other words a variable quantity. It is possible that some of these citations are not detected (citations in Chinese language journals, etc.). Therefore, we can take the number 248 as the lower limit of the citations of Prof. B. Zlatanov. For example,

from the submission of the dissertation to the writing of this review, the following additional citations appeared, which are not reflected in the candidate's citation list:

Applicant's Cited Publication: ***S. Karaibryamov, B. Tsareva, B. Zlatanov. Optimization of the Courses in Geometry by the Usage of Dynamic Geometry Software Sam, The Electronic Journal of Mathematics and Technology, Volume 7, Number 1, (2013) 22-51.***

Citations:

- J. C. Silva, Implementation of an Educational Software to Reinforce the Learning of Geometry and Measurement in High School Students. Escuela Superior Politécnica de Chimborazo, 2023.
- Samuel Boateng, PhD Thesis, An Investigation of Students' Learning of Integral Calculus with Maple Software and Paper-Pencil Strategies in the Western Region of Ghana, University of Agder, Kristiansand & Grimstad, Norway, 2022
- Sava Grozdev, Veselin Nenkov, Tatiana Madjarova, Poncelet-Gergonne Circle, Symmetric Polynomials and Baricentric Coordinates, International Journal of Computer Discovered Mathematics (IJCDM), Volume 7, 338–343, 2022
- Sava Grozdev, Veselin Nenkov, Tatiana Madjarova, Poncelet-Gergonne Circle of a Triangle, Moving Between Two Fixed Circles, International Journal of Computer Discovered Mathematics (IJCDM), Volume 7, 324–337, 2022

Applicant's Cited Publication: ***V. Ivanova, B. Zlatanov, Implementation of fuzzy functions aimed at fairer grading of students'tests, Education Sciences, Volume 9, Issue 3, September 2019, Article number 214***

Citations:

- Beyza Esin Özseven, Naim Çağman. A Novel Student Performance Evaluation Model Based on Fuzzy Logic for Distance Learning. International Journal of Multidisciplinary Studies and Innovative Technologies, 6(1), 29-37 (2022). DOI:10.36287/ijmsit.6.1.29
- Daniel Doz, Darjo Felda, Mara Cotič, Combining Students' Grades and Achievements on the National Assessment of Knowledge: A Fuzzy Logic Approach, Axioms, 11(8), Article number 359, 2022 (Web of Science, IF=1.824, Q2, SCOPUS, SJR=0.441, Q3)

Applicant's Cited Publication: ***V. Ivanova, B. Zlatanov, Application of Fuzzy Logic in Online Test Evaluation in English as a Foreign Language at University Level,*** AIP ***Conference Proceedings, 2172, Article number 040009, 2019***

Citations:

- Daniel Doz, Darjo Felda, Mara Cotič, Combining Students' Grades and Achievements on the National Assessment of Knowledge: A Fuzzy Logic Approach, Axioms, 11(8), Article number 359, 2022 (Web of Science, IF=1.824, Q2, SCOPUS, SJR=0.441, Q3)

We can also add to the scientific activity of prof. Zlatanov that he is a supervisor of a successfully defended PhD student at FMI of PU. I have information about Dr. Atanas Ilchev as I was a member of his SJ. The title of A. Ilchev's PhD thesis is: "On some classes of cyclic operators with pairs of points of the best approximation*".*

3. Basic quantitative and qualitative indicators of the activity of the candidate for the acquisition of a scientific degree

I will examine the applicant's activity in several aspects:

3.1. Publications for participation in the competition: The list of these publications includes 18 scientific articles.

Two of the specified 18 articles were "used" by Prof. B. Zlatanov in the competition for academic position of "Professor". These two publications (as mentioned by the author) are included in the peer-reviewed procedure only for the convenience of the reader. The techniques introduced therein are used substantially in two of the chapters.

I will comment on the remaining 16 articles. 14 scientific papers are printed, and two are in the printing process, which has not been completed at the time of writing this review. Three of the publications are in the scientific conferences proceedings (more precisely, in the

proceedings of MATTEX CONFERENCE PROCEEDING: 2018, 2020, 2022), and the remaining 13 are published in the scientific journals, some of which are indexed with impact factor. 8 of these 16 papers are published in the journals with an impact factor. All articles are referenced in WoS and/or Scopus and have an impact rank. We can indirectly judge the quality of these 13 scientific articles by the high classification of the journals in which they are published:

Classification of the journals	Number of publications	Summary indicator	Average indicator
Impact factor	8	$\Sigma_{IF} = 14{,}559$	IF=1,820
Q1 (JCR) (JCR - Journal Citation Reports)	4	$\Sigma_{SJR} = 3{,}001$	SJR=0,750 (SJR - Scimago Journal Rank)
Q2 (JCR)	4	$\Sigma_{SJR} = 1{,}462$	SJR=0,366
Q3 (JCR)	0	$\Sigma_{SJR} = 0{,}000$	SJR=0,000
Q4 (JCR) without IF	1	$\Sigma_{SJR} = 0{,}127$	SJR=0,127
AIP Conference Proceedings, without IF	3	$\Sigma_{SJR} = 0{,}554$	SJR=0,185
Web of Science; SCOPUS without IF and SJR	1	$\Sigma_{SJR} = 0{,}000$	SJR=0,000

Table 4

- The numerical data indicated in the table above deviate slightly from the data provided by the applicant.
- The reasons for these differences are:
- In my numerical data, the first two publications from the list of scientific publications on the topic of the dissertation are excluded, since as I said above, they are involved in the procedure for acquisition of academic position "Professor". The applicant may also have excluded these publications when calculating his asset in relation to meet the minimum national requirements;
- Two of these articles are published in the journal Axioms, which received an IF on July 28, 2022, and the procedural documents were submitted on July 8. For this reason, the author used their instant indexing in SCOPUS with SJR and WoS without IF;
- In my data presented, the publications that are in the process under printing are included. I am convinced that these papers will be published, and therefore I have not neglected them;
- The candidate did not consider that the journal: International Journal of Pure and Applied Mathematics indexed of Q4 and SJR=0.127 (in the year of publication of his scientific work).

I will indicate several authoritative scientific journals in which prof. Zlatanov has published his papers:

- *Applied Mathematics and Computation;*
- *Journal of Fixed Point Theory and Applications;*
- *Nonlinear Analysis: Modelling and Control;*
- *Fixed Point Theory.*

According to the number of authors, we can distribute the publications as shown in the next table:

Number of authors	Number of publications
1	4
2	6
3	3
4	2
5	1

Table 5

3.2. Citations of the candidate's scientific works: In the procedure for acquiring the scientific degree "Doctor of Sciences", the candidate participated with 20 citations (out of total 248). The distribution of these citations as follows:

- 19 citations from the articles published in journals indexed in Web of Science and Scopus databases;
- 1 citation is in a journal publication reported in Zentralblatt Math.

Eight publications are cited 20 times.

3.3. Reference to the fulfillment of the minimum national requirements: The fulfillment of the minimum national requirements by the candidate is shown in the following table:

National indicators	Minimum number of points	Materials submitted by the candidate	Points achieved by the candidate
AThesis for awarding the degree "Doctor"	**50**	PhD Thesis	**50**
B AThesis for awarding the degree "Doctor of Sciences"	**100**	PhD Thesis	**100**
G. Scientific publications (outside the DSc Thesis or its corresponding scientific publications)	**100**	Scientific publications in the journals indexed in WoS, Scopus, and Zentralblatt Math: Q1 4 publications x 75 p.=300 p.; Q2 4 publications x 60 p.=240 p.; Q3 0 publications x 45 p.= 0 p.; Q4 0 publications x 36 p.= 0 p.; SJR 4 publications x 30 p.=120 p.; Zentralblatt Math 1 publication x 18 p.= 18 p.; Total 678 p.	**678**
E. Citations in the scientific editions	**100**	Presented citations from the publications printed in the indexed journals: WoS and Scopus 19 citations x 8 p.=152 p.; Zentralblatt Math 1 citation x 4 p.= 4 p.; Total 156 p.	**156**
TOTAL	**350**		**934**

Table 5

I will give three remarks about the numerical data presented in this section:

Remark 1. In the last column of the row, indicator G of Table 5, the number of points (678) indicated here does not coincide with the points (618) declared in the relevant given reference, entitled Reference for the fulfillment of the minimum national requirements under the ADASRB. This discrepancy is due to the circumstance I described in section 3.1 of the review (immediately after Table 3). However, I will specify that the number of points indicated in the review is greater than the one presented by the author.

Remark 2. Only 20 citations are reported in indicator D in Table 5, 19 of which are in the journals indexed in WoS and Scopus, as well as one citation in a journal referenced by Zentralblatt Math. The remaining (noted) 228 citations are not reported in the reference for meeting the minimum requirements. This is probably due to the fact that 'lost points' are not necessary to satisfy the minimum national requirements.

Remark 3. It is clear from Table 5, that every single indicator of the minimum national requirements is exceeded by the applicant even if not all his achievements are taken into account. I will note that, the minimum required indicator points relating to the degree "Doctor of Science" have been met totally by the candidate more than twice.

PU and in particular FMI do not have additional minimum requirements for obtaining the scientific degree "Doctor of Sciences". This circumstance is due to the fact that the ADASRB canceled the presence of additional requirements of the universities for the acquisition of the educational and scientific degree "Doctor" and the scientific degree "Doctor of Sciences".

4. Relevance of the topic and appropriateness of the set goals and tasks

The important (and in many cases basic importance) of Banach's theorem for contraction operators in many mathematical sciences is known. Its role is fundamental both in a theoretical aspect (mainly in the acquisition of new theoretical knowledge as sufficient conditions for the existence of solutions to general mathematical problems or for the existence of certain qualities of these solutions) and in an applied aspect in finding the solutions or their approximations for specific mathematical problems (for example differential equations or differential inequalities). It can definitely be said that in some cases this theorem represents a basic method for solving various mathematical problems such

- Finding the solutions or their approximations to a wide range of equations and inequalities (algebraic, differential, integral, integro-differential, functional, etc.);
- Estimation of the error when replacing the unknown sought solution with specific approximations thereof;
- Determination of specific qualities of solutions of equations or inequalities such as periodicity, boundedness, asymptotic equivalence, etc.

Bringing a given problem (for example, one of the problems listed above) to the solution of abstracted equation of the form $Tx = x$, where T is a contraction operator in a suitably chosen metric space) is in many cases a difficult, and in some problems, an impossible task. Therefore, various generalizations of the fixed point method for contraction operators are sought. Finding such summaries continues to be a modern topical task, and it seems to me that this theme will remain "eternal".

The presented dissertation summarizes and enriches in some particular cases the inexhaustible topic of shrinking images and their corresponding fixed points and points of best approximation.

I will proudly point out that a significant group of Bulgarian researchers has been involved in the development of this part of mathematical analysis and the fundamental, and qualitative theory of the relevant differential equations. In this regard, the representatives of PU are in the forefront of Bulgarian mathematicians. I will mention the names of Petko

Proynov, Boyan Zlatanov, Snezhana Hristova, Andrey Zahariev, Hristo Kiskinov, Atanas Ilchev, Stoil Ivanov, ets. I may have missed some colleagues, but this was not done on purpose.

It is known that mathematics, and in particular mathematical analysis, develops not only because of the numerous applications in science and practice. Science develops for its own sake. The construction of many theories, the majority of which will "drop out" for objective reasons over time, is a way to preserve the vitality of knowledge and its aspiration for development.

Here, I will summarize my belief that the results in the dissertation are up-to-date and in sync with modern and classical trends in mathematical analysis. The place of the research presented in the dissertation will be determined by the interest it will generate in the mathematical community.

5. Problem understanding

I defend the opinion that each researcher (no matter how talented and able to work) is not able to get to know and assimilate all the published results on a given scientific topic (even in a relatively "narrow" scientific direction). The main reasons for this opinion are:

- Availability of relatively numerous sources containing information on the researched topic (on a global scale);
- Externally imposed limited access to information. For example, some of the sources are technically unavailable, others are financially unavailable, third sources are unavailable from a linguistic point of view;
- Existence of repetition or "excessive proximity" between some studies, due to which some of them are ignored by the user or by the scientific community;
- Lack of scientific interest on the part of the specific researcher to certain aspects of the theory (although related to his research), etc.

For the reasons stated above, knowledge of a given scientific problem should mean that the researcher has a certain amount of scientific information of the necessary quality (not all), which is sufficient to understand the content of the essential part of the scientific results on the subject. It also means that the researcher is able to carry out independent scientific interpretations of the results and subsequent scientific research.

I believe that Prof. B. Zlatanov knows in detail the current state (as well as the historical development) of the considered scientific problems and the corresponding mathematical objects of research in the scientific work submitted for review. I reach this conclusion by considering:

- There a serious, content-rich and fundamental introduction to the topic at the beginning of the thesis. Helpfully the main definitions and results of the leading authors on which the author's research is based are given.
- During the reading, even at initial acquaintance with the scientific work, it is not necessary to use additional, introductory, reference literature on the subject. In other words, the beginning of the thesis has the quality of a book for advanced students. This circumstance is convenient for the professional reader without prior knowledge on the topic of dissertation. In addition, the introduction shows that the author well know what has been achieved in the theory under study;

- The candidate shows high professional competence about fluently using the terminology, the main definitions and statements on the subject, the ability to combine specific properties and qualities of different mathematical objects;
- Sufficient concrete applications are presented, which once again convince us of the reliability and usefulness of the presented results;
- The indicated literature used and some comments on the papers of other authors represent a confirmation of the author's sympathy for the scientific problems under consideration. Here, I will note that the literature (or as it is customary to say "bibliography") to the considered thesis contains only scientific works that are directly related to the author's research. There are no literary sources that are placed "on the external merit";
- The many essential remarks and consequences, some of them of independent interest, clarify and complement the theoretical results of the author. All give me a reason to consider that the theory of these complex mathematical objects is deeply thought out;
- It can be seen that the author creates his own research technology in several places in the reviewed work. In addition, he is able to creatively transform known results and research methods of other authors. He possesses the ability to reasonably set optimal constraints on the objects under consideration, and in some cases to overcome difficulties of a technical nature.

I also answer the recently standard question regarding the originality of the results: *The dissertation lacks elements of repetition and plagiarism by foreign researchers.*

6. Research methodology

The main apparatus of conducting research in the thesis are the methods and some scientific facts from several mathematical sciences:
- Real mathematical analysis;
- Functional analysis represents the main apparatus in the formulations and proofs in the dissertation;
- Method of shrinking images in full metric spaces is used many times in the thesis. Because of its exceptional applications in scientific research in mathematics, I consider it out of functional analysis;
- Methods for partially ordered metric spaces;
- Theory of pairs of fixed points;
- Theory of triple fixed points
- Cyclic images, fixed points and points of best approximation;
- Ekeland's variational principle;
- Modular spaces and fixed points in modular spaces;
- Theory and basic tasks for oligopolistic markets, etc.

In conclusion of this section, I will emphasize that, as in almost all dissertations in mathematics, and here, there is no particular mathematical method (and only this method) that is applied to suitable objects in order to discover new facts. The application of different knowledge and methods, their combination in order to achieve new results, is the scheme by which the research in the peer-reviewed dissertation work is carried out. I will note that this way of uncovering new facts is difficult and inherent in researchers with extensive creative potential.

7. Content, characteristics, and evaluation of the thesis:

I will comment sequentially on the main parts of the dissertation.

Preface: It has a general informational nature. The main objects of research are indicated. Generalizations of Banach's fixed point theorem related to pairs of fixed points and points of the best approximation are reviewed, as well as some of their applications. The investigation is further developed with natural generalizations concerning triple fixed points and triple points of the best approximation, and semi cyclic representations of three variables. The most important results achieved by the author are highlighted.

Introduction: It is an important and integral part of the thesis. Extremely reasonable, convenient, accurate and in sufficient volume, the necessary preliminary facts are given: notations, definitions, theorems, consequences, sources, authors, etc., which are relevant to the next considerations. Through this information, created or developed by the author, the studied mathematical theories and their applications are presented professionally, following the principle of cause-and-effect relationships. In this way, the results are presented elegantly and at the same time comprehensible even for the reader who is not sufficiently prepared on the subject. Another goal is also achieved: the user of the results or the curious reader is fully convinced which of the presented results are the personal work of the applicant for the scientific degree or of the collective in which he participates.

As is well known, Banach's fixed point theorem constitutes a "base" for creating numerous theorems and algorithms for finding fixed points for various specific operators defined (in general) in various spaces with certain properties. Essentially, these newly obtained theorems and algorithms are found to be suitable for:

- Conditions related to the workspace X , using its specific characteristics;
- Conditions related to the shrinking operator T acting in space X , i.e. $T : X \to X$ based on its properties.

The goal is that the combination of these two types of conditions guarantees the existence of at least one fixed point ξ $(T\xi = \xi)$.

One of the generalizations of Banach's theorem that the candidate uses in his research belongs to G. Hardy and T. Rogers. Let denote by $\rho(x, y)$ the distance between arbitrary elements x and y from the complete metric space X . Then the operator $T : X \to X$ possesses a unique fixed point if the classical contraction operator inequality (known from Banach's theorem) is replaced by the more general inequality

$$\rho(Tx,Ty) \le \alpha\rho(x,y) + \beta\big(\rho(Tx,x) + \rho(Ty,y)\big) + \gamma\big(\rho(Tx,y) + \rho(Ty,x)\big),$$

where the real constants α, β, γ are positive and satisfy the inequality $\alpha + 2\beta + 2\gamma < 1$.

The concept of a pair of fixed points for a given operator was introduced by V. Lakshmikantham and his students in 1987. Let be given the operator $F : X \times X \to X$. An ordered pair $(x, y) \in X \times X$ is called a fixed point pair for F , if

$$x = F(x, y) \quad \text{and} \quad y = F(y, x).$$

Authors' studies from the group of V. Lakshmikantham on the existence and uniqueness of a pair of fixed points were initially carried out for operators in partially cone-ordered Banach spaces. Later, this concept was extended to the operators in partially ordered metric spaces (A. Ran and M. Reurings, 2004). The idea was further developed by V. Berinde and M.

Borcut for the triple fixed points. For this purpose, let the operator $F: X \times X \times X \to X$. An ordered triple $(x, y, z) \in X \times X \times X$ is called a fixed point triple for the operator F if the next equalities hold

$$x = F(x, y, z), \quad y = F(y, x, y) \quad \text{and} \quad z = F(z, y, x).$$

Subsequently, in 2016, W. Kirk, P. Srinivasan and P. Veeramani introduced and studied cyclic images and fixed points for such images. We will say that the operator $T: X \to X$ is cyclic if there exist sets $A, B \subset X$, for which we have

(i) $$T: A \cup B \to A \cup B,\ T(A) \subseteq B \text{ and } T(B) \subseteq A.$$

In the thesis, on p. 10^8, the definition of the cicle operator is unclear. I will clarify in advance that in the review I will use the symbol n^k, which means: line k of page n, counted from the top. The symbol n_k has a similar meaning - in this case, the numbering of the line is from the bottom. In the place indicated above is written:

$$T: A \subset X \subseteq B, \quad T: B \subset X \subseteq A.$$

Probably, the symbol „$\subseteq$“ should be replaced by the symbol „$\to$“. A further generalization is related to the concept of an ordered pair of cyclic images (F, G). In this case, the requirements are decisive

(ii) $$F: A \times A \to B \quad \text{and} \quad G: B \times B \to A.$$

It is possible, a cyclic image $T: A \to B$ to not posses a fixed point. Then, it is logical to look for that element $x \in A$, which in some sense is closest to its image Tx. Following this logic, A. Eldred and P. Veeramani in 2006 introduced the concept of points of the best approximation. The point $\xi \in A$ is called the point of the best approximation for the cyclic image T in A, if

(iii) $$\rho(\xi, T\xi) = dist(A, B) = \inf\{\rho(x, y):\ x \in A,\ y \in B\}.$$

If two sets A and B have a non-empty common part, then the point of the best approximation is a fixed point for the image T. A natural generalization of the definition mentioned above is the concept of a pair of points $(x, y) \in A \times A$ of the best approximation of the operator $F: A \times A \to B$. In this case it is assumed that

$$\rho(x, F(x, y)) = \rho(y, F(y, x)) = dist(A, B).$$

As the author notes, for the practical applications it is not enough to know that a fixed point exists for a given, specifically investigated operator. It is necessary to construct a procedure (usually iterative) to find approximately the fixed point (if it is unique). An important feature of approximate methods is finding an upper estimate of the a priori and/or a posteriori error. From there, the speed of convergence between the fixed point and the iterative series of approximations is determined. This issue has been given serious attention in the thesis and a sufficient amount of satisfactory research has been carried out.

The ideas of evaluating the errors of replacing the single points of the best approximation with elements of the corresponding convergent iteration series are further developed in modular function spaces. I will briefly describe this concept using the following remarks:

Remark 4: Let

- Ω is a nonempty set;
- M_∞ is a space of all extended measurable functions, i.e. all functions $f:\Omega\to[-\infty,\infty]$;
- $\rho: M_\infty \to [0,\infty]$ is a convex, even function.

Then ρ is a regular convex pseudomodular if:

- $\rho(0)=0$;
- ρ is monotone, i.e.

$$(\forall f,g\in M_\infty):\left(|f(\omega)|\le|g(\omega)|,\ \omega\in\Omega\right)\Rightarrow\rho(f)\le g(g);$$

- ρ is orthogonally sabaditive, i.e.

$$(\forall f\in M_\infty)(\forall A,B\in\Sigma,\ A\cap B\neq\varnothing)\Rightarrow\rho(f1_{A\cup B})\le\rho(f1_A)+\rho(f1_A),$$

  where Σ is sigma-algebra over Ω and 1_A is the characteristic function for set A;
- ρ possess the Fatou property, i.e.

$$(\forall f\in M_\infty)\left(\forall\{f_n\}:|f_n(\omega)|\uparrow|f(\omega)|,\ \omega\in\Omega\right)\Rightarrow\rho(f_n)\uparrow\rho(f);$$

- ρ is order continuous in E, i.e.

$$\left(\forall\{g_n\},\ g_n\in E\right):\left(|g_n(\omega)|\downarrow 0\right)\Rightarrow\rho(g_n)\downarrow 0,$$

  where with E the linear space of all simple functions with supports on a ring P.

Remark 5: ρ is a regular convex modular if it follows almost everywhere $f=0$ from equality $\rho(f)=0$.

Remark 6: Linear space $\left\{f\in M_\infty,\ |f(\omega)|<\infty\right.$ is called a modular functional space almost everywhere if $\lambda\to 0$ from $\rho(\lambda f)\to 0$.

The theory of modular spaces was initiated by H. Nakano in connection with the theory of ordered spaces, which idea was further developed by J. Musielak and W. Orlicz. The study of the geometry of modular function spaces started with the research of W. Kozlowski.

The so-called duopoly market is also discussed in the introduction of the dissertation. The considered mathematical model of the mentioned market corresponds to many natural assumptions, the most important of which are the following:

- Two (and only two) manufacturers (traders) compete in the market for the same customers;
- The goods produced by the producers are indistinguishable from a market point of view;
- The behavior of producers is rational, i.e. the objective of each of them is to obtain the maximum profit, provided that its competitor has a constant volume of production and maintains a constant price.

Under some natural additional assumptions, the model has the form

$$P(x+y)+xP'(x+y)-c_1'(x)=0;$$
$$P(x+y)+yP'(x+y)-c_2'(x)=0,$$

where

- x and y are the production quantities of the goods, respectively, of the first and the second producer. Therefore, the total output that is realized in the market is $x+y$;

- Function $P = P(x+y)$ represents the market price;
- Functions $c_1(x)$ and $c_2(x)$ show the costs of the first and second producers depending on the quantity of output produced.

Finally, I will note that the dipole model is an excellent illustration of the applications of fixed point pair theory.

Mainly, the research in the dissertation is focused on the concepts, statements, and a range of unsolved questions mentioned in the introduction.

First chapter: The question of the existence of pairs of fixed points in partially ordered metric spaces is investigated. Images that possess mixed monotone property are studied. Let's look at the image (operator) $F: X \times X \to X$, where $(X, \leq)$ is a partially ordered set. The image F possesses the mixed monotone property if the inequalities hold:

$$(x_1, x_2, y \in X,\ x_1 \leq x_2) \Rightarrow F(x_1, y) \leq F(x_2, y);\ (y_1, y_2, x \in X,\ y_1 \leq y_2) \Rightarrow F(x, y_1) \geq F(x, y_2).$$

The results are obtained by a modification and generalization of Ekeland's variational principle for images with the mixed monotonic property.

First, the basic assumption on the properties of the operator (corresponding to the properties of the contraction operator) is as follows:

$$(\exists \alpha = const \in [0,1)):\ (\forall x, y, u, v \in X;\ x \geq u,\ y \leq v) \Rightarrow$$

$$\rho(F(x,y), F(u,v)) + \rho(F(y,x), F(v,u)) \leq \alpha\rho(x,u) + \alpha\rho(y,v).$$

If there exists an ordered pair (x, y), such that $x \leq F(x,y)$, and $y \geq F(y,x)$, then there exists a pair of fixed points (x, y) of F . The conditions under which the fixed point is unique are indicated.

Another result in this chapter is related to an operator F with mixed monotone property having a contraction property, i.e. an inequality of the type

$$(\exists \alpha = const \in [0\ ,\ 1/2)):\ (\forall x, y, u, v \in X;\ x \geq y,\ y \leq v) \Rightarrow$$

$$\rho(F(x,y), F(u,v)) \leq \alpha\rho(x, F(x,y)) + \alpha\rho(u, F(u,v)).$$

Then, there exists again the fixed point (x, y) of the operator F (if $x \leq F(x,y)$ and $y \geq F(y,x)$). Uniqueness is guaranteed under the same requirements.

A similar result is obtained by the author of the thesis for an image with a mixed monotone property under the following shrinking constraint:

$$(\exists \alpha = const \in [0, 1/2)):\ (\forall x, y, u, v \in X;\ x \geq y,\ y \leq v) \Rightarrow$$

$$\rho(F(x,y), F(u,v)) \leq \alpha\rho(x, F(u,v)) + \alpha\rho(u, F(x,y)).$$

A summary of the results stated above in this chapter is given by a theorem about the image $F: X \times X \to X$, where $(X, \leq)$ is a partially ordered space. Assume that:

- $\left(\exists \alpha \geq 0,\ \beta \geq 0,\ \gamma \geq 0,\ \alpha + \beta + \gamma \in \left[0, \frac{1}{2}\right)\right):\ (\forall x, y, u, v \in X;\ x \geq y,\ y \leq v) \Rightarrow$

$$\rho(F(x,y), F(u,v))$$
$$\leq \alpha(\rho(x,u) + \rho(y,v)) + \beta(\rho(x, F(x,y)) + \rho(u, F(u,v))) + \gamma(\rho(x, F(u,v)) + \rho(u, F(x,y)));$$

- $x \le F(x,y)$ and $y \ge F(y,x)$.

Then there exists a fixed point (x,y) of the operator under consideration. Under certain conditions, the fixed point is unique. It is easy to see that the previous three statements are obtained from the last theorem by canceling two of the constants accordingly α, β, γ.

I will note that the question of the existence of other classes of operators with the mixed monotone property for which analogous results for the existence and uniqueness of pairs of fixed points hold, which are established by appropriate variational techniques, remains open.

Second chapter: The main goal in this chapter is to obtain "a priori" and "a posteriori" error estimates for the points of the best approximation. The technique for finding the ratings of the specified type for different classes of images is based on the author's research. I will take the liberty of describing in more detail the main result by which the grades were obtained: Let:

- $(X, \|.\|)$ is a uniformly convex Banach space;
- $A \subset X$ and $B \subset X$, A, B are closed and convex nonempty sets;
- $T: A \cup B \to A \cup B$ and T is a cyclic image (for the details, see (i));
- $(\exists k,\ 0<k<1): (\forall x \in A,\ \forall y \in B) \Rightarrow \rho(Tx,Ty) \le k\rho(x,y) + (1-k)dist(A,B)$.

Then:

- There is a single point of the best approximation for the operator T (for the details on the point of the best approximation, see (iii) of the review). Furthermore, the points of the best approximation satisfy the equalities:

  $$\xi = T^2\xi = T^4\xi = ... \in A \qquad \text{and} \qquad T\xi = T^3\xi = T^5\xi = ... \in B.$$

  The result belongs to A. Eldred and P. Veeramani;
- An a priori estimate is valid (the result is by B. Zlatanov):

  $$\left\|\xi - T^{2n}x\right\| \le \frac{\|x-Tx\|}{1-k^{2/q}}\left(\frac{\|x-Tx\|-d}{Cd}k^{2n}\right)^{1/q};$$

- An a posteriori estimate is valid (the result belongs to B. Zlatanov):

  $$\left\|\xi - T^{2n}x\right\| \le \frac{\left\|T^{2n-1}x - T^{2n}x\right\|}{1-k^{2/q}}\left(\frac{\left\|T^{2n-1}x - T^{2n}x\right\|-d}{Cd}k\right)^{1/q},$$

where $d = dist(A,B)$, C and q are specific constants relating to the convexity modulus properties of the Banach space $(X, \|.\|)$.

In the subsequent studies in this chapter, the author's technique is extended to image types with specific shrinkage properties. The images are research object by other authors (in the part with the existence and uniqueness of pairs of points of the best approximation). I will indicate the ratings obtained by the author for the image classes:

- Error estimation for pairs of the best approximation points for cyclic shrinking images (F,G) for which the shrinking property holds

  $$(\exists \alpha > 0, \beta > 0,\ \alpha + \beta < 1):\ (\forall (x,y) \in A \times A,\ \forall (u,v) \in B \times B) \Rightarrow$$

$$\rho\left(F\left(x,y\right),G\left(u,v\right)\right)\le\alpha\rho\left(x,u\right)+\beta\rho\left(y,v\right)+\left(1-\alpha-\beta\right)dist\left(A,B\right).$$

I would recommend Theorem 3.2 on p. 76 of the thesis (and also Theorem 8 on p. 18 of the abstract) to be edited so that to contain only the statements related to the values found by the author;

- Error estimation for pairs of the best approximation points for cyclic shrinking images $\left(F,G\right)$ for which the shrinking property holds

$$\left(\exists\alpha>0,\beta>0,\ \alpha+\beta<1\right):\ \left(\forall\left(x,y\right)\in A\times A,\ \forall\left(u,v\right)\in B\times B\right)\Rightarrow$$
$$\rho\left(F\left(x,y\right),G\left(u,v\right)\right)\le\alpha\rho\left(x,u\right)+\beta\rho\left(y,v\right).$$

Of interest is the theory of fixed points and points of the best approximation for ordered pairs of ordered pairs of images:

$\left(\left(F,f\right),\ \left(G,g\right)\right),\ F:A_1\times A_2\to B_1,\ \ f:A_1\times A_2\to B_2,\ G:B_1\times B_2\to A_1,\ \ g:B_1\times B_2\to A_2,$

where $A_1,A_2,B_1,B_2\subset X$. The following concepts are given:

- Cyclic shrinking ordered pair,
- Modified fixed pair of points,
- The best modified point pair approximation,
- Iteration rows, etc.

The main results refer to:

- Existence and uniqueness of pairs of points of the best approximation for a cyclic shrinking ordered pair;
- Existence and uniqueness of a fixed pair of points for a cyclic contracting ordered pair;
- Convergence of the iteration series to the corresponding elements of these points of the best approximation;
- Convergence of the iteration series to the corresponding elements of these fixed points;
- The priori and posteriori estimates of the approximation error from the iteration series.

The question of existence and uniqueness of the fixed point pairs and the best approximation point pairs is also considered for - p cyclic shrinking images. Corresponding the priori and posteriori estimates of the approximation errors are found.

A sufficient number of specific examples (usually systems of equations) illustrating applications of the main results are presented. I will note that in some cases the classic computer programs (Maple 18.00) are powerless.

Third chapter: The idea of points of the best approximation is enriched and developed for modular function spaces (for details, see Remark 6 of the review). Let ρ be a nonzero regular convex modular, L_ρ is a modular function space.

The modular distance between sets $A,B\subset L_\rho$ is defines as

$$d\rho\left(A,B\right)=\inf\left\{\rho\left(x,y\right);\ x\in A,\ y\in B\right\}.$$

Let the operator $T:A\cup B\to A\cup B$ be the cyclical image. The point $\xi\in A$ is called the point of the best approximation for that operator if $\rho\left(\xi,T\xi\right)=d\rho\left(A,B\right)$. The image T is called a cyclic shrinking image if

$$\left(\exists k\in\left(0,1\right)\right):\left(\forall x\in A,\ \forall y\in B\right)\Rightarrow\rho\left(Tx-Ty\right)\le k\rho\left(x-y\right)+\left(1-k\right)d\rho\left(A,B\right).$$

The following statement is valid. If:

- ρ is a uniformly continuous functional modular that has certain qualities (which I will not specify in the review);
- The sets $A,B\subseteq L_\rho$ are closed and convex, and $A\cup B$ is bounded;
- $T:A\cup B\to A\cup B$ is a cyclic shrinking image.

Then there exists a unique point $x\in A$, which is a point of the best approximation for the operator T in A, $T^2x=x$. The point of the best approximation is a limit of the series $\left\{T^{2n}x_0\right\}$ for an arbitrary initial approximation $x_0\in A$.

The ideas and approach to prove the above fundamental statement are used in the author's subsequent results related to in modular Orlicz function spaces. A function is called an Orlicz function if it is even, convex, continuous, nondecreasing on the interval $[0,\infty)$, $M(0)=0$ and $M(t)>0$ for $t\neq 0$. Let $L^0(\Omega)$ is the space of all measurable functions defined on the measurable space Ω, and which take real values. The Orlicz modular is defined as follows

$$\tilde{M}(f)=\int_\Omega M\left(f(t)\right)d\mu(t).$$

For the modular function space $L_{\tilde{M}}$, generated by M, the subsets A and B of $L_{\tilde{M}}$ and a cyclic shrinking image T, $T:A\cup B\to A\cup B$ are constructed so that the image has a single point of the best approximation in A. This point is limit for the series of even “powers” of the operator T, regardless of the initial approximation of A. I will note that the proposed limitations in the mentioned construction are quite numerous. It is difficult to trace their necessity. The presented example answers some of these questions, but it should be noted that the particular illustrative construction is (as should be expected) in a relatively simple case.

Among the contributions in this chapter, I will include the author's research on a pair of shrinking images in modular spaces. Under certain conditions, the question of existence and uniqueness of an ordered pair of points of the best approximation is answered positively: More precisely, if:

- ρ is a modular with specific properties;
- The sets $A,B\subseteq L_\rho$ are closed, bounded and convex;
- The operators $F:A\times A\to B$ and $G:B\times B\to A$ are a cyclic contracting pair (F,G).

Then

- There exists a unique ordered pair of points $(x,y)\in A\times A$ such that:

$$x=G\left(F(x,y),F(y,x)\right);\ x=G\left(F(y,x),F(x,y)\right);$$
$$\rho\left(x-F(x,y)\right)+\rho\left(y-F(y,x)\right)=2d_\rho(A.B).$$

(This means that (x,y) is the only ordered pair of the best approximation points for the operator F in the set A);

- $\left(F(y,x),F(x,y)\right)$ is a unique-ordered pair of the best approximation points for the operator G in the set B.

Modular spaces are studied and R. Kannan-type shrinking images defined in them. Let A be a nonempty subset of the modular function space L_ρ. An image $F: A\times A\to A$ is called shrinking of R. Kannan type if

$$\left(\exists \alpha = const,\ 0<\alpha<1/2\right):\ \left(\forall x,y,u,v\in A\right)\ \Rightarrow$$

$$\rho\left(F\left(x,y\right)-F\left(u,v\right)\right)\le\alpha\left(\rho\left(x-F\left(x,y\right)\right)+\rho\left(u-F\left(u,v\right)\right)\right).$$

Conditions for the existence and uniqueness of a fixed point for such images are found.

Let $A,B\subseteq L_\rho$. The operators $F: A\times A\to B$ and $G: B\times B\to A$ are called a Kannan-type cyclic contracting pair $\left(F,G\right)$ if

$$\left(\exists \alpha = const,\ 0<\alpha<1/2\right):\ \left(\forall\ \ x,y\in A;\quad \forall\ \ u,v\in B\right)\ \Rightarrow$$

$$\rho\left(F\left(x,y\right)-G\left(u,v\right)\right)\le\alpha\left(\rho\left(x-F\left(x,y\right)\right)+\rho\left(u-G\left(u,v\right)\right)\right)+\left(1-2\alpha\right)d_\rho\left(A,B\right).$$

It seems to me that on p. 173[7] in the thesis (and in the corresponding place in the abstract), the expression "$F\left(u,v\right)$" should be replaced by "$G\left(u,v\right)$" as written in the definition above of a contracting pair of Kannan-type operators

Let $\left(F,G\right)$ be a cyclic shrinking pair of Kannan-type images. Conditions for the existence and uniqueness of an ordered pair of points $\left(x,y\right)\in A\times A$, which are the best approximation for the operator F, are presented.

Also, $\left(F\left(y,x\right),F\left(x,y\right)\right)$ is the only ordered pair of the best approximation points for the operator G in the set B.

Fourth chapter: An extremely useful application of the fixed point pairs and point pairs of the best approximation of semicyclic images is found in the study of market equilibrium in duopoly markets.

Fifth chapter: In 2011, V. Berinde and M. Borcut introduced and explored the concept of triple fixed points: In the dissertation, the author summarizes this concept. Let $A_i,B_i,\ i=1,2,3$ be subsets of the metric space $\left(X,\rho\right)$ (six subsets in total). The ordered pair (each element of which consists of ordered triples) images $\left(F,G\right)=\left(\left(F_1,F_2,F_3\right),\left(G_1,G_2,G_3\right)\right)$ is called cyclic if

$$F_i: A\times A\times A=A^3\to B_i \text{ and } G_i: B\times B\times B=B^3\to A_i.$$

The point $\left(\xi_1,\xi_2,\xi_3\right)\in A^3$ is called a triple of fixed points for the ordered pair of ordered triples of images if $\xi_i=F_i\left(\xi_1,\xi_2,\xi_3\right)$. The point $\left(\xi_1,\xi_2,\xi_3\right)\in A^3$ is called the triple of the best approximation for the operator F, if

$$\rho\left(\xi_i,F_i\left(\xi_1,\xi_2,\xi_3\right)\right)=dist\left(A_i,B_i\right)=d_i.$$

An ordered cyclic pair of triples images $\left(F,G\right)$ is called a shrinking pair of type **one** if

$$\left(\exists\alpha_i^j=const\in\left[0,1\right);\ \alpha_1^j+\alpha_2^j+\alpha_3^j<1,\ i,j=1,2,3\right):$$

$$\left(\forall\left(x_i^1,x_i^2,x_i^3\right)\in A^3,\ \left(y_i^1,y_i^2,y_i^3\right)\in B^3\right)\Rightarrow$$

$$\sum\nolimits_{i=1,2,3}\rho\left(F_i\left(x_i^1,x_i^2,x_i^3\right),\ G_i\left(y_i^1,y_i^2,y_i^3\right)\right)\le\sum\nolimits_{i,j=1,2,3}\alpha_i^j\rho\left(x_i^j,y_i^j\right).$$

If in the above description of a contracting pair of type one, we replace the last inequality with the inequality

$$\sum_{i=1,2,3}\rho\left(F_i\left(x_i^1,x_i^2,x_i^3\right),\ G_i\left(y_i^1,y_i^2,y_i^3\right)\right)\le\sum_{i,j=1,2,3}\alpha_i^j\rho\left(x_i^j,y_i^j\right)+\sum_{j=1,2,3}\left(1-\alpha_1^j-\alpha_2^j-\alpha_3^j\right)d_j\,,$$

then, the ordered cyclic pair of triple images (F,G) is called a shrinking pair of type **two.**

The conditions for existence of a fixed point $(\xi_1,\xi_2,\xi_3)\in(A_1\cap B_1)\times(A_2\cap B_2)\times(A_3\cap B_3)$ for a generalized cyclically shrinking of a pair images (F,G) are presented in the thesis. The fixed point is identical for each of the operators F and G. A priori and a posteriori estimates of the error are found when replacing the fixed point coordinates with terms of the corresponding iteration series.

An important result wich summarizes the research of other authors is a set of sufficient conditions for the existence of the points of the best approximation for each of the elements of a generalized cyclically shrinking pair of images (F,G) type two, defined in a convex Banach space. A priori and a posteriori estimates of the error when replacing the coordinates of the fixed points with members of the corresponding iteration series are found.

As an application of the results in this chapter, the existence, and uniqueness of a market equilibrium in a three-firm oligopoly is studied.

8. Contributions and significance of the development in the science and practice

As written above, the contributions in the thesis are formulated correctly by the candidate in the Conclusion to the dissertation. Contributions can be attributed to the thematic enrichment and extension of the methods of fixed point, and their applications. Extensions and generalizations of known theorems and methods are proposed which are useful in proving working theorems about the existence of the fixed points and the points of the best approximation in several types of spaces. These results are illustrated “richly” with concrete, and diverse applications. For me, this fact is a very important and convinces me of the usefulness of the author's research.

It is difficult to list all the contributions in the dissertation work, therefore I will focus only on those that more seriously challenge my attention:

- The conditions of existence and uniqueness of the pairs of fixed points for classes of images with the mixed monotone property are derived. The developed technique is based on the generalizations of Ekeland's variational principle;
- The estimates of error for pairs and triple points of the best approximation using iterative approximation methods are found;
- It is proved for the pairs of fixed points or points of the best approximation (x,y) that they must satisfy the condition $x=y$;
- Modified cycled images and modified point pairs are defined. Possibilities for the applications of this class of images in solving non-symmetric systems of equations are indicated;
- Pairs of the points of the best approximation in modular function spaces are investigated. Their application for solving systems of equations is illustrated;

- A new class of images called semi-cyclic images is defined. Conditions for the existence and uniqueness of pairs of fixed points for semi-cyclic images are found;
- The question of existence and uniqueness of market equilibrium in duopoly markets is investigated. The proposed technique allows a significant shortening of the restrictive classical conditions of the market model (for example, the need for differentiability of the profit function is eliminated).

In my opinion, the research in the thesis is sufficiently deep and important that it will occupy a worthy and lasting place in science.

9. Assessment of the candidate's personal involvement in the dissertation publications

The thesis is based on 18 scientific publications with the participation of the candidate, of which I accept 16 for review (two of the publications were used in previous procedures, therefore I accept them for reference). As I noted in Table 4, there are 4 independent publications, and the rest are with co-authors.

The applicant's participation in the research work of these joint publications may be considered equivalent to the work of the other co-authors. I have no further information to refute the statement made. I will explicitly emphasize that the candidate's publication activity related to the dissertation work is more than satisfactory. Publications are sufficient as a number and the most are published in reputable scientific journals which are thoroughly covered by various secondary databases. This means that they are easily discovered and monitored by the interested scientific community. This circumstance, in my opinion, will lead to the "wide" use of the results and their even more active citation in the near future.

10. Abstract

The abstract fully complies with the requirements of the Rules for the Development of the Academic Staff of the PU, as well as the ADASRB and the Regulations for the Implementation of the ADASRB. There are specified in it:
- Content of the dissertation. A replay has been made in the content in section "Approbation of the obtained results" (see p. 4_2, and p. 4_3 in the Abstract, and also in p. 7_3, and p. 7_4 in the thesis);
- Introduction containing basic definitions, statements, techniques that are used in the research work on the topic of dissertation);
- Sequentially, following in the exposition by chapters in the work submitted for review, the main definitions and concepts are given;
- Consecutively, according to the chapter-by-chapter presentation in the dissertation, the essential limitations and the resulting main and auxiliary statements obtained by the author are formulated;
- Basic applications of the theoretical results what are based on specific examples;
- Main contributions of the dissertation work;
- Author publications on the topic of the dissertation;
- Approbation of the presented results;
- Bibliography.

The material in the abstract is presented in such a way that the reader can gain a complete and adequate idea of the results in the dissertation.

I will pay attention to the fact that the numbering of theorems, lemmas, remarks, definitions, etc. in the abstract differs from the numbering of their originals in the thesis. This makes to match the records of the two documents difficult. In addition, the full list of the bibliography from the dissertation should be transferred without abbreviations to the abstract for the reasons stated above, although the abstract may not cite all the literature sources.

11. Critical remarks and recommendations

I have no significant critical notes and comments that could change my positive opinion about B. Zlatanov's dissertation work. Indeed:

- The topic of the thesis can be defined as contemporary. Although the scientific community has devoted “resource” and serious “attention” to this powerful mathematical apparatus for more than a century, the fundamental ideas of fixed points remain among the most intensively developing areas of mathematics and its applications;
- The specific tasks, realizing the goals of the works proposed for review, have been selected and solved sequentially in a natural order and successfully;
- The proves, although not so easy to understand, we can “summarize” as complete and presented in sufficient academic detail (although in places it is rather “tightly” written). We can conclude that insight into the statements and their corresponding evidence requires good preparation and serious attention of the reader;
- The independent style of the author's work is clearly visible;
- Some concrete realizations of statements (for example, related to popular spaces, sets, and operators) are cleverly framed as stand-alone statements in the form of theorems, applications, and corollaries;
- Further understanding of the conditions can be achieved by carefully following the example implementations provided.

12. Personal impressions

I have known Prof. B. Zlatanov for ten years, on the occasion of the following joint activities:

- Several common participations in scientific juries (including that of his doctoral student);
- Several accreditations of the specialties and doctoral programs in FMI of the PU. I was supervisor of the procedures on the part of NAOA, and Prof. B. Zlatanov as deputy dean of FMI.

In my opinion, B. Zlatanov is a well-established scientist and university teacher with the experience for more than 20 years. I assume that, the colleagues from FMI at PU share my opinion, and are familiar with a number of his other positive qualities of him (possibly some negative ones as well). For this reason, Prof. B. Zlatanov was elected deputy dean (I think for a second term).

I consider his personal human qualities as a “wealth” what to be respected and emulated. I will point out some of them:

- Militant, uncompromising honesty;
- Openness;

- Taking personal responsibility and defending justice.

Someone may say that I'm “adding a little salt to the sleeve”, but I will answer: “I say what I see”.

Conclusion

The thesis contains scientific and scientific-applied results that represent an original contribution to science. The submitted documents and studies meet all the requirements of the ADASRB, the Regulations for the implementation of the ADASRB and the relevant Regulations of the PU “Paisii Hilendarski” for the acquisition of the scientific degree “Doctor of Sciences”.

On the basis of the above, I confidently give my positive assessment of the conducted research, presented in the above-reviewed dissertation work, abstract and scientific publications.

I propose to the honorable Scientific Jury to award the scientific degree “Doctor of Sciences” to Prof. Dr. Boyan Georgiev Zlatanov in the Professional Field of Higher Education: 4. Natural Sciences, Mathematics and Informatics; Professional direction: 4.5. Mathematics (Mathematical Analysis).

01/09/2022

Reviewer: .....

(Prof. DSc Angel Dishliev)