REVIEW

from Vasil Georgiev Angelov, D. Sc., Ph. D.,
Professor at the University of Mining and Geology "St. Ivan Rilski"

on a dissertation work for the acquisition of the academic degree **"Doctor of Sciences"**
at Plovdiv University "Paisii Hilendarski"

Field of higher education 4. Natural Sciences, Mathematics and Informatics;
Professional field 4.5. Mathematics;
Mathematical Analysis

Author: **Pgrofessor Boyan Georgiev Zlatanov, PhD, Plovdiv University "Paisii Hilendarski"**

Subject: **Application of coupled fixed points and coupled best proximity points**

1. Subject of the referee

By order № RD-21-1333 of July 18, 2022 of the Rector of Plovdiv University "Paisii Hilendarski" (PU) I was appointed a member of the scientific jury to provide a dissertation defense procedure on a topic **„Applications of coupled fixed points and coupled best proximity points“** to acquire of the academic degree **„Doctor of Sciences“** of Plovdiv University in the field of higher education 4. Natural Sciences, Mathematics and Informatics, professional field 4.5. Mathematics, Mathematical Analysis. The author of the dissertation is professor Boyan Georgiev Zlatanov –Department of Mathematical Analysis, Faculty of Mathematics and Informatics, Plovdiv University “P. Hilendarski”.

The set of paper materials presented by Prof. Boyan Zlatanov is in accordance with the Regulations for the Development of the Academic Staff of the University of Plovdiv and includes all necessary documents as follows :

- Application to the Rector of PU “P. Hilendarski” to open a procedure for defense of the dissertation;
- CV;
- a copy of the diploma for the educational and scientific degree "doctor";
- protocols from departmental councils related to the opening of the procedure and the preliminary discussion of the dissertation work;
- Dissertation paper;
- Author's reference of the dissertation;
- list of scientific publications on the subject of the dissertation;;
- copies of scientific publications;
- declaration of originality and authenticity of the attached documents;
- certificate of compliance with the minimum national requirements.

The candidate has attached 18 publications for the doctoral dissertation.

2. Brief biographical data

Boyan Zlatanov graduated from FMI at SU "Kl. Ohridski" (1991-1996). In 2001, he became a doctor in Mathematical Analysis at the FMI of the PU "P. Hilendarski" with a dissertation on "Geometric properties of some classes of Banach spaces with an unconditional basis", in 2008 – associate professor, and in 2019 – professor.

3. Actuality of the topic and appropriateness of the set goals and problems

The relevance of the present dissertation is predicated on its subject matter. Fixed point theory has many different applications and has become one of the main methods of nonlinear analysis. The dissertation deals with generalizations of the Banach shrinking image theorem related to the coupled fixed points and their applications. Generalizations of Ekeland's variational principle are reviewed. A technique is proposed to prove existence results for coupled fixed points for images with mixed monotone property using the generalization of the variational principle.

Results for existence-uniqueness of coupled points of best proximity points by finding the error estimates in successive approximations are obtained. The possibilities of finding exact solutions of systems of equations using the enriched theory of fixed point pairs are illustrated. The notions of coupled fixed points and coupled best proximity points in modular function spaces are summarized. A new class of images is defined that is distinct from both cyclic and non-cyclic images. This class, called semicyclic representations, arises naturally in the study of market equilibrium in duopoly markets. Conditions for existence-uniqueness of coupled fixed points for semicyclic images are found. Models of duopoly markets are constructed using semicyclic representations that substantially generalize the classical theory of duopoly markets. The obtained results are illustrated with different models. The results are summarized for tripled fixed points and tripled best-proximitypoints, as well as for semicyclic representations of three variables that naturally arise modeling markets dominated by three players. All these facts speak for the relevance of the problems posed and their applications.

4. Knowing the problem

The author has many articles devoted to this topic, which shows a serious insight into the issue.

5. Research methodology

The chosen research methodology allows achieving the set goal and obtaining an adequate answer to the tasks solved in the dissertation work.

6. Characterization and evaluation of the dissertation

The dissertation consists of an Introduction, five chapters and References.

I note that the numbering of the literature below is according to the author's reference.

Chapter I is entitled "Coupled Fixed Points in Partially Ordered Metric Spaces". It contains some generalizations of known results [11], [26] related to coupled fixed points in partially ordered complete metric spaces. A generalization of Ekeland's variational principle is made, which is used in the proofs of the existence of coupled fixed points of images with mixed monotone property in partially ordered metric spaces.

In Chapter II “Estimation of Error for Coupled Best ProximityPoints” a technique is developed to obtain an estimate of the error of best approximation for coupled best proximity points for some classes of images based on results from [51]. An error estimate is found for coupled fixed points and coupled best proximity points for cyclic shrinking images. Applications have been made to solve: integral equations of the Fredholm type in the space of summable square functions, linear algebraic equations, transcendental equations, etc. I consider the examples of solving two linear algebraic equations with two unknowns to be unsuccessful. At the end of Chapter II, generalizations are made for coupled best proximity points for *p*-cyclic images with application to solving one particular system.

Chapter III "Coupled Best Proximity Points in Modular Functional Spaces" summarizes the idea of coupled best proximity points in modular functional spaces and gives applications for integral operators in Orlich spaces with a functional modular. The main results are reached by going through a generalization of several nodal lemmas in the study of best proximity points in modular functional spaces. The notion of "best proximity point for cyclic shrinking images" has been transferred from metric spaces to modular function spaces. A series of auxiliary results are proved which lead to the first result for best proximity points in modular function spaces – Theorem 3.1. Next, an application of the best proximity points in modular Orlich spaces is made for a class of integral equations. Then theorems for coupled fixed points and coupled best proximity points in modular function spaces are proved (Theorem 18 and Theorem 19, respectively). Applications of the proven theorems are also given.

In Chapter IV, "Application of Coupled Fixed Points and Coupled Best Proximity Points of Semicyclic Images to the Study of Market Equilibrium in Duopole Markets," various applications of the apparatus developed so far are discussed. It begins with an examination of the described duopoly market model and the conditions for its equilibrium. Concepts from previous chapters are summarized in order to apply them to existing models. The notion of coupled fixed points for semi-cyclic images is introduced. Error ratings found. A model of a market with two competing firms, each producing a single good, with the goods being interchangeable, is considered. A linear

and non-linear model are investigated. Various generalizations of these models are given, which are obtained by introducing more factors, which deepens the author's research.

Chapter V "Tripled Fixed Points and Tripled Best Proximity Points" summarizes the concepts from the previous chapters. Special attention is paid to generalize cyclic collapsing pair of image triples, to semicyclic ordered image triples, and to the existence-uniqueness of a market equilibrium in a three-player oligopoly.

7. Contributions and significance of the development for science and practice

I generally accept the author's claims for the contributions made and summarize them as follows:

I. Ekeland's variational principle is generalized for images with mixed monotone property in order to find conditions for the existence-uniqueness of coupled fixed points for classes of images with mixed monotone property. The classes of problems for which coupled fixed points exist have been extended. The obtained error estimate for coupled best proximity points was used to find the error estimate for coupled and tripled best proximity points.

II. It is proved that for the cyclic images considered so far, the coupled fixed points or coupled best proximity points (x, y) must satisfy the condition $x = y$. The concept of ordered pairs of cyclic images is generalized, defining a new type of images and points called modified cyclic images and modified coupled points, respectively. This new class of images can also be used to solve non-symmetric systems of equations. Examples are given of finding exact solutions to systems of transcendental equations for which the approximate methods used in Maple 18.00 could not find an exact solution.

III. The notion of best proximity point in modular function spaces is summarized. Using possible convexity modulus generalizations in modular function spaces, generalizations of the key results of Eldred and Veermany are proved. The technique for exploring best proximity points in modular function spaces, developed in [50], has been applied to exploring coupled best proximity points in modular function spaces. Examples are given for solving systems of equations for which the approximate methods of Maple 18.00 cannot find an exact solution.

IV. The concept of an ordered pair of semi-cyclic images is introduced, which naturally arises in the study of market equilibrium in oligopolistic markets. A new model is presented to study the existence and uniqueness of market equilibrium in duopoly

markets, which is based on response functions. Its advantages over the classical model for maximizing profit functions are shown, by eliminating the need for differentiability, exploring along the contour of the sets of possible productions and obtaining conditions for the stability of the process from successive changes of productions.

V. A possibility of generalization of some of the problems studied in Chapters 2-5 on tripled fixed points, tripled best proximity points and their application to the study of oligopoly markets with three players using semicyclic representations of three variables is considered.

In general, a good impression is made by the various applications in economic tasks.

8. Assessment of dissertation publications

Publications can be classified by type (articles – 12 issues; reports – 6 issues), by significance (articles in publications with an impact factor – 8 issues, of which 3 are Q1, 5 – Q2), by place of publication (articles in refereed international journals – 12 issues, reports in proceedings of international scientific conferences abroad – 3 issues; reports in proceedings of international scientific conferences in Bulgaria – 3 issues), by number of co-authors (independent – 6 issues; with one co-author – 6 issue; with two co-authors – 3 issues; with three or more co-authors – 3 issues). All articles are in English.

9. Personal contribution of the author

The general presentation of the results in the dissertation shows a unity that does not raise doubts about the main personal participation of the dissertation author in the conducted research, as well as the conclusion that the formulated contributions and obtained results are his personal merit.

10. Author's reference

The author's reference was made according to the requirements of the Regulations of the PU "P. Hilendarski" and reflects the main results achieved in the dissertation work.

11. Critical remarks and recommendations

I believe that too much attention has been paid to the application of the methods obtained in the dissertation for solving linear algebraic systems with two unknowns. This facilitates research, which is definitely applicable to non-linear systems as well.

12. Personal impressions

I have known Boyan Zlatanov for several years professionally and I have good impressions of his results and scientific interests.

13. Recommendations for future use of dissertation contributions and results

I recommend Prof. Zlatanov to continue his research.

CONCLUSION

The dissertation contains **scientific, scientific-applied and applied results that represent an original contribution to science and have no elements of plagiarism**. They meet all the requirements of the Law on the Development of the Academic Staff in the Republic of Bulgaria (ZRASRB), the Regulations for the Implementation of ZRASRB and the relevant Regulations of the PU "P. Hilendarski". The presented thesis fully corresponds to the minimum national requirements adopted in connection with the Regulations of the PU for the application of the ZRASRB. The dissertation work shows that Boyan Georgiev Zlatanov has in-depth theoretical knowledge and professional skills in the scientific specialty Mathematical Analysis, demonstrating qualities and skills for conducting research with obtaining original and significant scientific contributions.

Therefore, I give my **positive assessment** of the conducted research, presented by the above-reviewed dissertation work, abstract, achieved results and contributions, and I propose to the respected scientific jury to award the scientific degree "Doctor of Sciences" to Boyan Georgiev Zlatanov in the field of higher education 4. Natural sciences, mathematics and informatics, Professional direction 4.5. Mathematics (Mathematical Analysis specialty).

29.08. 2022 **Reviewer: .....**

Professor Vasil Angelov, D.Sc.