REVIEW

by Prof. Andrey Ivanov Zahariev, PhD –
Faculty of Mathematics and Informatics, Plovdiv University 'Paisii Hilendarskii'
of a dissertation for the award of a scientific degree
"Doctor of Science" in Plovdiv University 'Paisii Hilendarskii'
in Area of Higher Education: 4. "Natural sciences, Mathematics and Informatics"
Professional Field: 4.5 Mathematics (Mathematical Analysis)

Author: Prof. Dr. Boyan Georgiev Zlatanov member of the Faculty of Mathematics and Informatics of Plovdiv University "Paisiy Hilendarskii"

Title: "Applications of Fixed Point Pairs and Best Proximity Point Pairs"

1. Subject of review

According to order number PD-21-1333 from 18.07.2022 of the Rector of the Plovdiv University "Paisiy Hilendarskii" (Plovdiv University) I have been appointed as a member of the scientific jury to ensure a procedure for the defense of a dissertation work with title "Applications of pairs of fixed points and best proximity point pairs" for obtaining the scientific degree "Doctor of Sciences" of the PLOVDIV UNIVERSITY in the field of higher education 4. "Natural sciences, mathematics and informatics" professional direction 4.5 "Mathematics" (Mathematical analysis). The author of the dissertation work is Prof. Dr. Boyan Georgiev Zlatanov - Department of "Mathematical Analysis" at the Faculty of Mathematics and Informatics of Plovdiv University "Paisiy Hilendarskii".

The set of paper materials presented by Prof. PhD Boyan Georgiev Zlatanov is in accordance with Article 45 (4) of the Regulations for the Development of the Academic Staff of the PLOVDIV UNIVERSITY , includes the following documents:

1. Request to the Rector of the Plovdiv University to disclose the procedure for the defense of a dissertation work;

2. Curriculum vitae in European format;

3. Protocols from departmental councils related to the opening of the procedure and the preliminary discussion of the dissertation work;

4. Summary;

a) in Bulgarian;

b) in English;

5. Lists of publications;

a) list of scientific publications on the subject of the dissertation;

b) list of scientific publications used in the procedure;

c) a list of all scientific publications;

6. lists of citations;

a) a list of noticed citations in the publications of the dissertation work;

b) A list of noted citations used in the procedure;

c) A list of all noticed citations;

7. Dissertation work;

8. Certificate of compliance with the minimum national requirements;

9. Declaration of originality and authenticity of the attached documents;

10. Copies of the publications on the topic of the dissertation work;

11. Diploma for educational and scientific degree "PhD" - original - No. 27519, 27.08.2001, Higher Attestation Commission;

The candidate has attached a list of 18 (eighteen) publications, of them 16(sixteen) in journals and 2 in conference reports. The dissertation is based on 16(sixteen) of them and the points for compliance with the National minimum requirements are obtained only by summing the points of the articles numbered 1-12 and 15-18 (16 pcs.). As the author has correctly noted, the points from the articles under number 13 [129] and number 14 [130] are not included in the Reference for compliance with the National minimum requirements for obtaining the scientific degree "Doctor of Sciences" under 4.5. Mathematics such as articles 13 [129] and 14 [130] are listed only for the convenience of the reader.

From the 16 pcs. publications indexed in WoS and SCOPUS are 10 (of which 7 in journals with an impact factor (four journals with quartile Q1, three journals with quartile Q2), two in SCOPUS with SJR and one in WoS and SCOPUS without IF and SJR). Two of the publications have been accepted and are in the process of being printed, and one publication will be indexed in WoS SCOPUS too. A total of 20 citations are indicated, of which 19 are in journals referenced in WoS and SCOPUS and one in Zbl.

2. Brief biographical data.

Prof. Dr. Boyan Georgiev Zlatanov graduated in the period 1991-1996 from SU "St. Kliment Ohridski", Faculty of Mathematics and Informatics, as professional specialist "Mathematics" with specialization - "Mathematical Analysis"; "Mathematics and Informatics teacher", 5-year year course of study - equivalent to a "master's" degree, and in 2001 he received the PhD degree. From 1999 until now, the candidate has been a lecturer at the Plovdiv University 'Paisii Hilendarskii', occupying successive academic positions as assistant, professor assistant and associate professor and professor. He was deputy dean of the branch of "Paisiy Hilendarskii" PLOVDIV UNIVERSITY in the city of Smolyan and is the deputy dean of FMI of PLOVDIV UNIVERSITY "Paisiy Hilendarskii" at present.

3. Actuality of the topic and appropriateness of the set goals and tasks.

It is well known that fixed point theorems based on and/or induced by the classical Banach theorem for shrinking maps are one of the most powerful tools used in research in the field of nonlinear analysis and its multiple applications. In this aspect, all possible types of generalizations of Banach's theorem on shrinking images represent an "evergreen topic" (always an actual topic) for research. The main directions for generalizations related to the present study are the following: Let

X be a complete metric space with metric ρ and **T**:X→X be an operator. One direction for generalizations relevant to the present exposition is by appropriately introducing additional structures in the metric space such as partial ordering, convexity and etc., depending on the area of application of the results sought. Another direction of generalization is through a suitable modification of the classical condition for a contraction satisfied by the operator **T**. The dissertation cited a number of modified types of contractions such as the images of Kannan [72], Chaterjea [28], Hardy–Rogers [53], and others. Even a cursory look at the publication activity in these directions is enough to convince us that the topic chosen by the author is current and significant.

4. Knowing the problem

The author has excellent knowledge of the state of research in the field treated in the dissertation, the basis for this conclusion is the extensive and well-chosen bibliography on the subject, as well as the thorough and analytically written introduction giving a thorough and balanced overview of the subject. The numerous results obtained by the author in various directions on this topic demonstrate his creative and thorough approach to the analysis and evaluation of the publications of other authors..

5. Research methodology

An analysis of the results of the publications presented by the author included in the dissertation work, as well as the dissertation work itself, convince us that he not only masters the techniques developed by other authors in their research on this topic, but is also the creator of methods (techniques) suitable for research on this topic. These techniques of his allow him to solve not only the problems posed and investigated in the dissertation work but also the results obtained with their help find a number of applications in other fields of science such as economics, which is one of the essential goals of the presented work.

6. Characterization and evaluation of the dissertation work

The presented for a review The Dissertation "Fixed Point Pairs Applications and Best Proximity Point Pairs" contains 314 pages and consists of a preface, introduction, five chapters, conclusion and bibliography. The bibliography contains 133 sources. The list of author publicationson the dissertation consists of 16 titles.

Generalizations of Banach's classical fixed point theorem related to the existence and existence of pairs of fixed points, pairs of points of best proximity, as well as a number of their applications are considered in the dissertation work. For maps with the mixed monotone property and the sets generated by them, the author has proved a generalization of the classical Ekeland variational principle, which, generally speaking, cannot be applied in a number of important applications. A technique is also proposed to prove existence results for pairs of fixed points for maps with the mixed monotone property using the resulting generalization of the variational principle. Results for the existence and uniqueness of pairs of points of best proximity with are refined and developed by

finding the error estimates using sequence of successive iterations. It has been proven that if the classical error estimation approach developed by the author in [129] is used for the pairs of fixed points or points of best proximity (x; y) , they must necessarily satisfy the condition x = y. Generalizations of the notions of fixed point pairs, best proximity point pairs, and ordered pair of cyclic maps are proposed that allow the ordered pair (x; y) to consist of two distinct points. As an application of the obtained results, an approach is proposed to reduce the problem of solving systems of equations to problems of existence (calculating) of pairs of fixed points or points of best proximity. The possibilities of finding exact solutions to systems of equations using the refined and further developed theory of fixed point pairs are illustrated. In addition, generalizations of the concepts of fixed point pairs, and best proximity point pairs in modular functional spaces are defined, and the possibilities for solving systems of equations using cyclic maps in modular functional spaces, which arise from the studied system of equations. A new class of maps is introduced, which are different from both cyclic and acyclic maps, and which are called by the author semicyclic maps. The motivation for introducing this new class of maps is that this class of maps arises naturally in the study of market equilibrium in duopoly markets. Sufficient conditions for the existence and uniqueness of pairs of fixed points for semicyclic maps are obtained. As an application of the studies of semi-cyclic maps, the author constructs with their help a model of a duopoly market, which model substantially generalize the classical theory of duopoly markets. The obtained theoretical results are illustrated with various model examples. The ideas of fixed-point pair generalizations are further developed with triple-fixed-point and best-proximity triple-point generalizations, including semicyclic representations of three variables that naturally arise in modeling three-player-dominated markets.

In the thorough and analytically written introduction, the author has provided all the necessary definitions and facts of functional analysis mainly related to the partially ordered metric spaces, uniformly convex Banach spaces, the modular spaces necessary for the understanding of the studied problems, as well as a thorough and balanced overview of the main statements of leading authors on selected subject areas related to the researched topics presented in the dissertation work.

Chapter 1 is devoted to obtaining sufficient conditions for the existence and uniqueness of pairs of fixed points in partially ordered complete metric spaces and is based on the Publications[4, 59, 131]. The results obtained in §1.2. are based on a nontrivial generalization of Ekeland's variational principle for maps with the mixed monotone property introduced in §1.1. In §1.3. a meaningful example illustrating the results obtained in §1.2 is given. Using the generalization of Ekeland's variational principle in partially ordered, complete metric spaces in §1.4 and §1.5, the existence of pairs of fixed points for maps with the mixed Chatterjea-type or Hardy–Rogers-type monotonic properties is proved. If, in addition, every ordered pair of elements of the space X considered as an element of the Cartesian product X × X is bounded from above or below, then the pair of fixed points is unique.

Chapter 2 is based on the publications[58, 60, 63, 133] and three problems are addressed in it. First, the error estimation problem for pairs of fixed points and pairs of best proximity points for cyclic shrinking maps in uniformly convex Banach spaces is considered (§2.1 and §2.1). I will note that the first result on "a priori" and "a posteriori" error estimates for points of best proximity for cyclic maps in uniformly convex Banach spaces was obtained by the author of the thesis in [130]. The results obtained in §2.2 give a priori and a posteriori error estimates for pairs of fixed points (Theorem 2.5) and pairs of best proximity points for cyclic shrinking maps (Theorem 2.4). As the second topic in this chapter, modified pairs of fixed points and pairs of points of best proximity are investigated, and the obtained results in §2.3 are essentially related to the reduced application for solving arbitrary linear systems. The third topic is discussed in §2.4, which builds on [54]. In it, sufficient conditions for the existence and uniqueness of pairs of fixed points and points of best proximity for p–cyclic shrinking maps, where p is a natural number, are obtained. In chapter 2, a large number of illustrative examples are given, which show the applicability and effectiveness of the obtained results.

Chapter 3 is based on the publications[64, 65] and investigates the fixed point pair problem and the best proximity point pair problem for cyclic ρ-shrinking maps in modular function spaces. More precisely, the author has made a suitable generalization of the idea of points of best proximity in modular functional spaces, which has allowed him to construct applications for integral operators acting in Orlich functional spaces equipped with the Orlich functional modular. In §3.1. the author presents a non-trivial generalization for the case of modular function spaces of the classical lemmas of A. Eldred and P. Veeramani (given as Lemma 1 and Lemma 2 in the introduction). Paragraph §3.2. is devoted to the generalization of the notion of points of best proximity in metric spaces introduced in the classic work of A. Eldred and P. Veeramani [42], for the case of modular function spaces, and in it, sufficient conditions for existence and uniqueness are obtained of the ρ-point of best proximity. The obtained result (Theorem 3.1) is applied in studying the existence and uniqueness of ρ-points of best proximity for classes of integral operators important for the applications of modular Orlich functional spaces. Paragraph 3.3 is built on the basis of the publication [65]. Sufficient conditions for the existence and uniqueness of pairs of fixed points (Theorem 3.3) and pairs of best proximity points (Theorem 3.4) for ρ-contracting maps in modular function spaces are derived there. In §3.4, sufficient conditions for the existence and uniqueness of pairs of points of best proximity (Theorem 3.5) are derived for the important special case of ρ–Cannan shrinking maps in modular function spaces.

Chapter 4 is devoted to the application of pairs of fixed points and pairs of points of best proximity to the semicyclic representations introduced by the author in the study of market equilibrium in duopoly markets. The author has launched the idea that essentially one of the ways to find the equilibrium production (x; y) can be considered as the task of finding a pair of fixed points of the response functions F and f introduced in [47] for the two producers involved in the duopoly market. This approach allows him to combine and generalize the classical models of Cournot and Bertrand, which enables him to study the most general case when the two participants in the duopoly market

can compete simultaneously on prices and quantities. In §4.1. the necessary definitions related to the semicyclic maps introduced by the author are given in §4.2 sufficient conditions for the existence and uniqueness of pairs of fixed points (theorem 4.1), "a priori" and "a posteriori" error estimates are obtained, as well as the convergence rate of the iteration process is estimated. Section 4.3. is devoted to the applications of the obtained results in the various cases of a duopoly market, including the most general case where the two participants in the duopoly market can compete simultaneously on prices and quantities. Section 4.4. is based on the publication [68] as the results obtained in §4.2. also established for the important case semicyclic maps of Hardy–Rogers type for the case of general metric spaces. The obtained results are applied in the study of a duopoly market in the cases of a special form of the price function and a generalized reaction function. In paragraph §4.5. is a generalization of Ekeland's variational principle (Theorem 4.4) for semicyclic maps with the mixed monotone property. We will note the fact that Ekeland's variational principle is not applicable to cyclic and semicyclic maps that satisfy the mixed monotonic property, the results of which were used in the conducted study of the existence of market equilibrium in duopoly markets. In § 4.6, sufficient conditions for the existence of at least one pair of fixed points for multivalued semicyclic representations are obtained, which result enables the author to explore a more realistic model in which firms can choose for their production program from among a set of possible productions, i.e. response functions that are multivalued maps are considered. Section 4.7 contains sufficient conditions for the existence and uniqueness of pairs of best proximity points for shrinking semicyclic maps of type two, "a priori" and "a posteriori" error estimates (Theorem 4.6), as in the case for points of best proximity the rate of convergence remains an open problem. The obtained results are applied to the consideration of a duopoly in the case when the intersection of the sets of possible productions of the two participants has an empty section (this is the case for example when the participants produce different but interchangeable goods).

Chapter 5 is devoted to the study of the generalization of the notion of triple fixed points introduced in [20] and triple points of best proximity introduced in [120], as the author used the ideas from his publication [133]. It contains sufficient conditions for the existence and uniqueness of triples of fixed points (Theorem 5.1) and of triples of best proximity (Theorem 5.2) for contracting semicyclic maps of type two (a cyclic contracting pair of triple maps), "a priori" and a posteriori error estimates, as well as the convergence rate of the iterative process was evaluated. The obtained results are applied to the consideration of oligopoly in the case of three participants and sufficient conditions guaranteeing market equilibrium for the oligopoly of three participants are obtained.

7. Contributions and significance of the development for science and practice

After familiarizing myself with the author's research presented in the dissertation presented to me for review, I find that the main goals formulated in the preface of the dissertation have been achieved.

The main theoretical contributions of the author presented in the dissertation work are non-trivial generalizations of the classical Banach fixed point theorem, related to the existence and

uniqueness of fixed point pairs, best proximity point pairs, as well as a number of their mathematical applications related generally speaking, with the solution of different types (algebraic, transcendental, etc.) systems. The most important of them are:

1. The author has proved a generalization of Ekeland's classical variational principle for maps with the mixed monotone property. It should be noted that the generalization made is non-trivial and allows covering new classes of maps for which Ekeland's variational principle cannot be applied. This allows him to obtain sufficient conditions for existence as well as conditions for uniqueness of pairs of fixed points for classes of maps with the mixed monotone property and to extend the classes of problems for which pairs of fixed points exist.

2. It should be emphasized that the first result on "a priori" and "a posteriori" error estimates for points of best proximity for cyclic maps in uniformly convex Banach spaces was obtained by the author of the thesis in his publication [130]. This technique is developed and adapted in the dissertation and used to find an error estimate for pairs and triples of best proximity points for cyclic shrinking maps in uniformly convex Banach spaces.

3. A generalization of the notion of ordered pairs of cyclic maps is introduced by defining a new type of maps and points, called modified cyclic maps and modified point pairs, respectively. The results obtained as a consequence of the introduced generalization are non-trivial and important given the fact that for the cyclic maps considered so far, the pairs of fixed points or the pairs of points of best proximity (x; y) must satisfy the constraint x = y, which does not allow applications in this case of non-symmetric systems. The advantage of the introduced new class of maps is that it can also be used to solve non-symmetric systems of equations. Theoretical problems are illustrated with examples of finding exact solutions to systems of transcendental equations for which approximate methods used in the Maple release 18 algebraic computer system cannot find the exact solution.

4. On the basis of different variants (types) of generalizations of convexity modulus in modular functional spaces, the author in [129] proved generalizations of the fundamental lemmas of A. Eldred and P. Veeramani Veermani . In the dissertation, using the obtained result in [129], the notion of points of best proximity in modular function spaces is summarized and a technique for studying points of best proximity in modular function spaces is created. The created technique is applied in the study of pairs of points of best proximity in modular function spaces. An application to solving systems of equations is introduced, and with an example is illustrated the case for which the approximate methods used in the Maple release 18 algebraic computer system cannot find the exact solution.

The main scientific and applied contributions of the author presented in the dissertation are:

A. The concept of an ordered pair of semi-cyclic maps is introduced, which naturally arises in the study of market equilibrium in oligopolistic markets. A new model is presented to study the existence and uniqueness of market equilibrium in duopoly markets, which is based on reaction functions, which model unifies and generalizes the classical models of Cournot and Bertrand. The resulting new model makes it possible to study the most general case when the two participants in the duopoly

market can compete simultaneously on prices and quantities. The advantages of the new model over the classical model are illustrated with examples that demonstrate its advantages in maximizing profit functions, by eliminating the need for differentiability of response functions, and in contour exploration of the sets of possible productions with a view to obtaining conditions for stability of the production process under successive changes.

B. A summary of some of the studied problems for triple fixed points, triple points of best approximation, and their application in the study of oligopoly markets with three players using semicyclic representations of three variables is made.

One of the merits of the dissertation work is the author's clearly outlined perspectives for research development given in the conclusions of each chapter.

8. Evaluation of Publicationson the dissertation work

In the competition for obtaining the scientific degree "Doctor of Sciences", the candidate participated with a total of 16 scientific papers, 14 of which were published in journals, all written in English, which we can classify as follows:

• 7 articles that are referenced and have an impact factor (four of them in journals with quartile Q1 and three in journals with quartile Q2).

• 3 articles that are referenced and indexed in WoS and SCOPUS with SJ; 2 published and 1 in process of printing,

• Articles that are referenced in WoS and SCOPUS without SJR – 2 items; 1 published and 1 in process of printing.

• Publication in conference proceedings – 2 items;

Of these works, 4 pcs. are without authors (3 of them have an impact factor), 6 pcs. are with 1 co-author, 2 pcs. have 2 co-authors, 3 pcs. have 3 co-authors, 1 pcs. has 4 co-authors.

There is no doubt that the works presented were not used in the candidate's previous competitions, since all the works mentioned above were published after the competitions for the acquisition of the scientific and educational degree "doctor" and the occupation of the academic position "associate professor" and were not included in the competition for "professor". Again, I emphasize that the two articles [129,130] from those listed in the list of 18 articles that were used by Prof. B. Zlatanov in the competition for the academic position of "professor" are included (as the author correctly mentions only for the relief of the reader, because the entered in these techniques are used substantially in two of the chapters) and the points of them are not used in the Reference for compliance with the National minimum requirements for obtaining the scientific degree "Doctor of Sciences".

The indicated numerical data exceed almost 2.5 times the minimum requirements specified in the requirements of the Law on the Development of the Academic Staff in the Republic of Bulgaria (ZRASRB), the Regulations for the Implementation of the ZRASRB and the relevant Regulations of Plovdiv University "Paisiy Hilendarskii".

I will note that the Hirsch h-index of the candidate for the science degree, calculated according to the Plovdiv University blish or Perish program, is **11**. A result that, for the mathematical sciences, is serious. The stated facts give me reason to claim that what Prof. PhD Boyan Zlatanov has achieved meets his claims.

9. Personal involvement of the author

My assessment of the author's personal involvement in the research conducted is that it is prevalent in every aspect of the research.

10. Summary

The summary contains 64 pages and meets the requirements of the relevant regulatory documents - the Law on the Development of the Academic Staff in the Republic of Bulgaria (ZRASRB), the Regulations for the Implementation of the ZRASRB, and the relevant Regulations of Plovdiv University "Paisiy Hilendarskii". It contains all the main results obtained in the dissertation work and it reflects sufficiently fully and accurately the content and the main contributions of the author. The important results (in the form of theorems and corollaries) are only stated without exposition of the relevant proofs. The logically connected informative nature of the summary makes a good impression. This circumstance gives a complete and clear idea of the researched problems and achievements in the dissertation. In this sense, the summary is also useful for specialists who are not familiar with the dissertation work in detail.

11. Critical remarks and recommendations

I have no significant critical notes and comments beyond my personal style and layout preferences, which of course may not be covered by the style and layout of the dissertation. I believe that this type of notes (personal preference type) should not be the subject of notes and comments, so I ignore them in the review.

12. Personal impressions

I know the author well with whom I have worked and continue to work as a colleague in the Department of Mathematical Analysis. He is a conscientious, hardworking and responsible colleague, respected in his collegiate community. The same shows himself as a tolerant and erudite teacher. I can confidently state that he is a world-class built scientist.

13. Recommendations for future use of dissertation contributions and results

I definitely consider that the dissertation's achievements are important and essential, and in this aspect I recommend that the perspectives outlined by the author for the development of research given in the conclusions to each chapter be implemented in subsequent research.

CONCLUSION

The results obtained in the dissertation work and the comments made above in the review give me reason to draw the following conclusions:

1. The dissertation contains serious theoretical studies related to the existence and uniqueness of pairs of fixed points, pairs of points of best proximity, as well as a number of their applications. These results are an original contribution of the dissertation student and are of serious scientific interest.

2. The dissertation contains applied mathematical models that illustrate the importance of the author's theoretical results;

3. The achievements in the dissertation work meet the requirements of the Law on the development of the academic staff in the Republic of Bulgaria (ZRASRB), the Regulations for the implementation of ZRASRB and the relevant regulations of Plovdiv University "Paisiy Hilendarskii" for the acquisition of the scientific degree "Doctor of sciences".

Due to the facts mentioned above, I give the following assessment of the research in the dissertation work:

The dissertation contains scientific and scientific-applied results, which represent an original contribution to science, as **I have not detected the presence of plagiarism** and meet all the requirements of the Law on the Development of the Academic Staff in the Republic of Bulgaria (ZRASRB), the Regulations for the Implementation of ZRASRB and the relevant Regulations of PLOVDIV UNIVERSITY "Paisiy Hilendarskii". The presented materials and dissertation results exceed almost 2.5 times the minimum national requirements, adopted in connection with the Rules of the Plovdiv University for the application of the ZRASRB.

The dissertation shows that Prof. PhD Boyan Georgiev Zlatanov possesses in-depth theoretical knowledge and professional skills in the scientific specialty "Mathematical Analysis" by demonstrating qualities and skills for conducting research with obtaining original and significant scientific contributions.

Due to the above, **I confidently give my positive assessment** of the conducted research, presented by the above-reviewed dissertation work, summary, achieved results and contributions, and I propose to the Honorable scientific jury to award the scientific degree "Doctor of Sciences" to Prof. PhD Boyan Georgiev Zlatanov in the field of higher education: 4. "Natural sciences, mathematics and informatics", professional direction: 4.5 "Mathematics" (Mathematical analysis).

24/08/2022   Reviewer: ………………………………….

 Plovdiv   (prof. PhD Andrey Zahariev)