

**University of Plovdiv "Paisii Hilendarski"
Faculty of Mathematics and Informatics**

DEPARTMENT OF METHODOLOGY OF MATHEMATICS AND INFORMATICS EDUCATION

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**Formation of creative thinking in students in teaching geometry
using dynamic geometric software**

SUMMARY

for awarding PhD degree

Research area:

1. Pedagogy;

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1.3 Methodology of training in ...

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The PhD thesis is discussed and directed for defence at an extended meeting of the Department of Methodology of Mathematics and Informatics Education. of the Faculty of Mathematics and Informatics at Plovdiv University "Paisii Hilendarski" on September 10, 2021.

The dissertation "Formation of creative thinking in students in the teaching of geometry using dynamic geometric software" contains 188 pages (173 pages of body text and 15 pages of appendices). It consists of a preface, 3 chapters, a conclusion, a bibliography, applications (4 in number). The bibliography contains 73 sources. The list of author's publications on the dissertation consists of 6 items.

The public defence of the PhD thesis will take place on at in the meeting hall of the New Building of Plovdiv University "Paisii Hilendarski", Plovdiv, 236 Bulgaria Blvd.

The PhD documents are available to those interested in the Dean's Secretary of the Faculty of Mathematics and Informatics, New building of Plovdiv University "Paisii Hilendarski", office 330, every working day from 8:30 to 17:00.

Contents

General characteristics of the PhD thesis	5
A subject of the research	5
An object of research	5
Objectives of the study	6
A working hypothesis	6
Basic tasks of the PhD thesis	6
Research methods	6
A summary of the PhD thesis	6
Introduction	8
Features of specialized dynamic geometric software <i>Sam</i> and <i>GeoGebra</i>	8
Defining infinite elements and the function <i>Swap finite and infinite points</i> in the dynamic geometric software <i>Sam</i> and <i>GeoGebra</i>	9
Connected figures	9
Chapter I System of tasks suitable for solving using DGS Sam	11
1.1 A Model of teaching through DGS in the study of the middle segment of a triangle	11
1.2 Application of DGS in the study of the centroid point of a triangle	13
1.3 Development of creative abilities in students by studying the circumscribed quadrangle in a dynamic environment. Generalization of the task from the 53rd International Mathematical Olympiad, Argentina 2012	14
1.4 Stimulation of creative activity in bilingual students through dynamic geometric software	17
Chapter II Research-oriented training with the use of DGS GeoGebra	17
2.1 GeoGebra to develop students' creative thinking in geometry classes	17
2.2 One generalization of the geometric problem from 19th Junior Balkan Mathematical Olympiad, Serbia 2015	18
2.3 Intersections of polyhedrons with a plane	19
2.3.1 The method of intersection planes	20
2.3.2 The projection method makes	21
2.3.3 An improvisation on Problem 22, when the parallelepiped is replaced with a pyramid	22
Chapter III Diagnostic procedure for control of students' knowledge and skills	23
3.1 Theoretical statement	23
3.1.1 Analysis of the curriculum in mathematics in the 11 form (second level)	23
3.1.2 Verification and assessment of knowledge through tests	23
3.2 Research program	24
3.3 Presentation and analysis of the results of the diagnostic procedure	24
A Study of the characteristics of the test	25
Processing of the results	25
Conclusion	26
Summary of the obtained results	26
Contributions of the PhD thesis	27
Prospects for future research development	28
List of publications included in the PhD thesis	28

Approbation of the obtained results	28
The connection between the contributions, the tasks, the paragraphs in the PhD thesis and the included publications.	29
References	29

General characteristics of the PhD thesis

Nowadays, the concept of mathematical competence includes the ability to work with modern software systems. Students and pupils develop their programming and algorithmic thinking skills and thus their digital competence goes beyond that of the traditionally passive user of information technology. On the other hand, it deepens mathematical knowledge and conceptual understanding of mathematics Kenderov (2010).

Dynamic geometry software (DGS) are a valuable helper for any of their users. They provide active participation in the gaining of knowledge, allow the learner to become a co-discoverer. With the help of DGS, pupils and students observe and discover relations, formulate and reject hypotheses, understand deeper the essence of the studied topic.

Let us mention the words of Whitehead, transmitted in his presidential address to the Mathematical Association of England in 1916 year: "Let the main ideas which are introduced into a child's education be few and important, and let them be thrown into every combination possible. The child should make them his own, and should understand their application here and now in the circumstances of his actual life. From the very beginning of his education, the child should experience the joy of discovery..." (The Aims of Education). Confirmation of his words is the experiment mentioned in (Zlatanov, 2018): "We conducted an experiment with several seven-year-old children with basic computer literacy. Our experience with children of this age is small, episodic and not in the classroom. They started drawing and using infinite elements and the *Swap End and Infinite Dots* function. It was very interesting that they did not ask why there are finite and infinite points, but they just used them in their drawings. They implemented the *Swap End and Infinity Points* function and enjoyed the results. Intuitive understanding of the infinite elements, discovering the relationships between the figures, peering into the world of dynamic drawings are good initial conditions for developing an interest in mathematics".

Using *GeoGebra*, infinite points and the swapping of finite and infinite points in (Tsareva & Zlatanov, 2016), an innovative method for teaching the topic "Intersection of a polyhedron with a plane" was presented. In this method the drawing work is significantly optimized and allows to generate solutions of whole groups of tasks, only by solving one of them with the help of a specially written applet *Swap finite and infinite points*. The applet can be used at any stage of the solution in presenting the solution step by step. Thanks to the powerful potential of *GeoGebra*, this approach expands the teaching of "geometry with a computer" and introduces a new creative component in it.

Part of the overall process of implementing dynamic geometric software in mathematics education is the personal choice of each teacher in applying an appropriate model for the practical application of ICT. This process also takes into account the related specific objective and subjective problems, those that relate to the educational system in this case, that of mathematics. The accumulated pedagogical experience is shared, researched and realized through the pedagogical experiment, presented in the present PhD Thesis, aiming at the evaluation of the efficiency of application of dynamic geometric software in the teaching of mathematics.

The above unequivocally argues the relevance of this dissertation, which is focused on creating a system of tasks and methodological tools for teaching mathematics with the application of dynamic geometric software on topics from general education mathematics curricula for 5, 8, 9 and 11 form.

A subject of the research is the educational environment with the application of DGS in the study of the topic "Intersection of a polyhedron with a plane" in the teaching of geometry to students from 11 form, as well as the impact of the system of tasks to increase creative thinking in geometry classes on topics of 5, 8 and 9 form.

An object of research are the knowledge, skills and, as far as possible, the competencies of

the students in the teaching of the students in geometry by using the DGS as a tool for increasing the quality for mastering the teaching material in mathematics.

Objectives of the study:

- A development of methodological tools and technology for teaching geometry using DGS
- Realization of a pedagogical experiment for diagnostics of the results of the training on the topic "Intersection of a polyhedron with a plane".

A working hypothesis - through the developed didactic environment and technology of learning with the use of DGS can be formed a creative thinking in students.

Basic tasks of the PhD thesis is to find appropriate didactic and methodological approaches and tools to be used to compile a system of geometry problems for the use of DGS in classes for 5, 8, 9 and 11 form of the Bulgarian secondary school, which will optimally fit into the generally practised model and to realize the increase of the creative thinking in the students.

The following **tasks of the PhD thesis** are formulated for realization of the set main goal:

1. To choose appropriate software tools to increase creative thinking
2. Development of a system of tasks through which to achieve the set goal
3. Justification of the methods, didactic techniques and their impact on students in mathematics education with the application of DGS
4. Conducting a pedagogical experiment in order to test the main hypothesis formulated in the present PhD thesis, analysis and interpretation of the results, formulation of conclusions.

Research methods

The methodological approach used to carry out the research is a diagnostic procedure for the control of knowledge, skills and to some extent competencies. The following methods were used to achieve the goals and objectives in the present study and to test the hypothesis:

- Research of pedagogical, psychological, methodological and educational literature related to the issues of the present dissertation research, for building the theoretical basis of the research
- Theoretical and empirical research methods, such as observation, comparison, analysis, synthesis, modelling, theoretical summaries, group discussions, talks with current mathematics teachers, tests
- Use of accumulated personal experience in the teaching of mathematics and in the teaching of mathematics with application of DGS in school
- Didactic experiment
- Mathematical and statistical methods for processing experimental data
- Using tools, including a system of tasks, tests and exams, to test students' knowledge and skills.

A summary of the PhD thesis

The PhD thesis is written on 186 pages and consists of an introduction, three chapters, a conclusion, a bibliography and four appendices.

The **Introduction** describes the main characteristics and specifics of the software used *Sam* and *GeoGebra*, which are used in solving the system of geometry problems. For conducting quality training through the use of DGS mandatory conditions are the availability of technical support for each student and prior acquaintance with the software products that will be used to implement this methodological approach to training. An overview and analysis of the tools used *Swap finite and infinity points*, as well as *Connected figures*. Other related concepts, theorems and theoretical formulations are also explained. It is justified why these topics are considered and taught through these tools.

Chapter I presents the implemented activities on the topics:

- Midpoint segment of a triangle studied at 8. form
- Centroid Point of a Triangle studied at 8. form
- Circumscribed Quadrangle studied at 8. form
- Geometric shapes studied at 5. form with bilingual students.

While the first two topics present systems of tasks that students largely experiment and implement on their own, in the other two topics DGS Sam is mostly used for demonstration by the teacher with subsequent independent experimentation by students. The topic *Circumscribed Quadrangles* ends with a summary of a task from the 53 International Mathematical Olympiad, Argentina, 2015, born with the DGS tools *Sam* and aimed at students with –deep interests in the field of mathematics in order to enrich their opportunities for creative research. The last topic shows that the proposed methodological approach is applied, researched and gives good results in the specific learning environment of bilingual students.

In **Chapter II**, after the introductory part, two topics are discussed: a task from the 19 Balkan Youth Mathematical Olympiad, Serbia, 2012, and the topic *Section of a polyhedron with a plane* is fully developed, which is also the aim of the research. The logical connection of these topics is the use of DGS *GeoGebra*.

The task of the 19 Youth Balkan Olympiad is to create completeness in the scope of the research using DGS *GeoGebra* in solving problems with students with increased interest in mathematics.

The arguments for the choice made are clarified:

- Methodology and technology for organizing and conducting the teaching of mathematics using the proposed concept and the selected methodological tools
- Specific software application for integration in the process of teaching mathematics.

Chapter III is devoted to the diagnostic procedure, explaining the theoretical, methodological and experimental formulation.

The arguments and details related to the planning and organization of the pedagogical experiment for evaluating the effectiveness of the proposed technological model are specified. The selected experimental groups and the package of criteria and indicators for evaluation of the learning outcomes of the students from the experimental groups are described.

The conducted pedagogical experiment is described. The results of the research have been processed by statistical means, an appropriate analysis has been performed, conclusions have been formulated from a test of the working hypothesis.

The **Conclusion** presents the achieved results, formulates the main contributions, reports and publications on the results of the PhD thesis and presents some perspectives for future development.

Applications includes: Questionnaire - students - bilinguals; Test; The expert evaluation survey; The results of the testing.

Introduction

Features of specialized dynamic geometric software *Sam* and *GeoGebra*

To meet the needs of the times in which we live - building people with solid knowledge and competencies (including mathematical knowledge and technological skills), a small team of the Faculty of mathematics and Informatics at the University of Plovdiv "Paisii Hilendarski" (authors of citep ZTs3, ZTs4 and student Samet Karaibryamov, working as a programmer after graduating from the Faculty of mathematics and Informatics at the University of Plovdiv "Paisii Hilendarski"), sharing the opinion expressed in (Kortenkamp & Dohrmann, 2010) ("software for educational use should be limited to bare needs") creates a specialized DGS *Sam* for interactive training of students of synthetic geometry. The main part of the program is contained in the two diploma theses of Samet Karaibryamov (Karaibriamov, 2010, 2011). The work started in 2009. The creative participation of students in the learning process (Zlatanov & Tsareva, 2014b) and especially the fact that for Samet Karaibryamov learning in synthetic geometry is a continuous research and creative process as well as the works (Grozdev et al., 2014; Grozdev & Nenkov, 2012a,b, 2013, 2014, 2015; Trifonova et al., 2013; Grozdev & Nenkov, 2017) affirm the view that learners at all levels of the education system are tempted by new technologies more than anyone else. DGS *Sam* has two versions *SamSchool* offering a dynamic drawing environment (Zlatanov et al., 2012a; Karaibriamov et al., 2012, 2013) and *SamFull*, which contains *SamSchool* and a module for intersecting polyhedra.

As it is known, *GeoGebra* is a well-developed universal dynamic geometric software and of course with greater capabilities than the specialized DGS *Sam*. A particularly useful advantage of *GeoGebra* is the preservation of structures after changes in object definitions in the "Object properties" window. The introduction of the operator *Swap finite and infinity points* in *GeoGebra* (Tsareva & Zlatanov, 2016) undoubtedly enriches it, which is evident from (Tsareva & Zlatanov, 2016; Zlatanov, 2014, 2017).

The drawing tools 17 in number are presented in a horizontal bar at the top of the screen Figure 1.

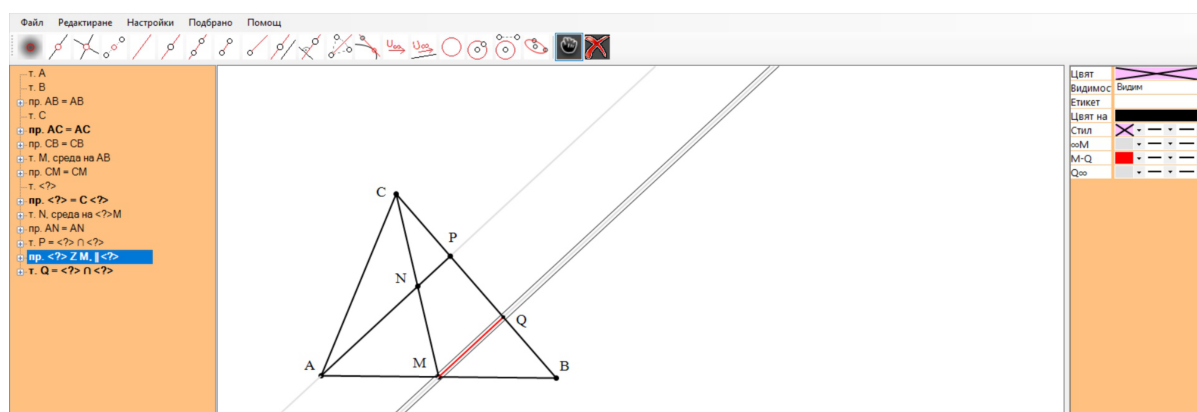


Figure 1. Interface DGS *Sam*

One of the advantages of DGS *Sam* is the possibility to present the solution "step by step" or

at stages chosen by the user. This is done easily through the panel on the bottom left. Initially the construction is staged, and then at the request of the user, after clicking on the rectangle labelled "Stages" to the construction are added - comment (explanation) concerning the current stage of the drawing; images.

Defining infinite elements and the function *Swap finite and infinite points* in the dynamic geometric software *Sam* and *GeoGebra*

DGS *Sam* includes an infinite point in its tools, and in its menu the special function *Swap finite and infinite points*. The new elements of DGS *textitSam* make it possible to generalize tasks and to create new ones.

The abstract axiomatic structure of projective spaces can be realized in different models. The most common model is the Euclidean model, where most drawings are performed. This model was created by expanding Euclidean space by introducing infinite elements. This model allows and stimulates the application of projective methods in solving problems of Euclidean geometry. Often these solutions are significantly shorter and direct their user to natural generalizations of the considered problems (Zlatanov, 2018).

Definition 1. *Two lines are said to intersect at an infinite point, provided that they are parallel.*

Under the guidance of one of the authors of DGS *Sam*, Assoc. Prof. Bistra Tsareva conducted an experiment in High School of Mathematics in Plovdiv by Bistra Taneva (teacher of informatics and information technology) with 5th grade students. In information technology classes, students are introduced to the world of DGS *Sam* and *GeoGebra* (Taneva, 2015a,b).

An empirical approach has been chosen to introduce the concept of infinite point. Although it is new for students and is not included in the curriculum, the results of the survey with a single question and three possible answers show that most students have mastered the new concept. An additional explanation was provided with the other students so that everyone would be ready for its application.

Connected figures

Construction 1.1 (Karaibriamov et al., 2013) We choose the arbitrary finite points A, U, V and two arbitrary infinite points U_∞ and V_∞ (Figure 2a). We do not use the „parallel lines“ tool. We construct the lines AB and AD as lines through the points AU_∞ and AV_∞ . We choose the points $B \in AU_\infty$ and $D \in AV_\infty$. We construct the lines BC and DC as lines through the points BV_∞ and DU_∞ and construct $C = BV_\infty \cap DU_\infty$.

Definition 2. *The parallelogram $ABCD$ (Figure 2a) constructed according to Construction 1.1 is called a universal parallelogram.*

Construction 1.2 (Karaibriamov et al., 2013) Let the base $ABCD$ be a universal parallelogram (Definition 2) with infinite points on its sides U_∞, V_∞ (Figure 2a).

We choose another arbitrary endpoint W and another arbitrary infinite W_∞ . We construct the lines $AW_\infty, BW_\infty, CW_\infty$ and DW_∞ , on which the surrounding edges will lie. Thus the free points are: $A, U, V, W, U_\infty, V_\infty$ and W_∞ . The vertex A' is an arbitrary point on the line. AW_∞ .

Following (Karaibriamov et al., 2013) we introduce the following notion:

Definition 3. *The parallelepiped $ABCDA_1B_1C_1D_1$, constructed according to Construction 1.2 is called a universal parallelepiped.*

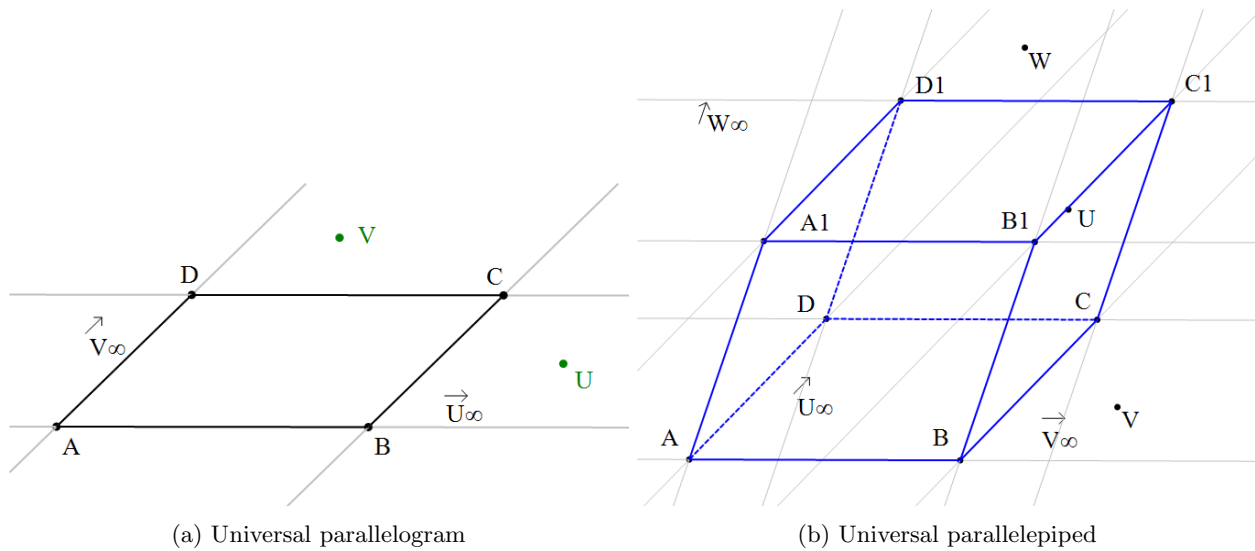


Figure 2

Definition 4. *The figures obtained from each other by applying the Swap finite and infinity points function are called connected figures.*

Examples of connected polyhedra are illustrated.

The next basic construction that we will use concerns the topic "Intersection of a polyhedron with a plane". It was proposed by the authors of (Tsareva & Zlatanov, 2016) because it ensures that by solving a specific problem from an intersection of the universal parallelepiped with a plane to obtain the solutions of a set of problems for the bodies connected with it, as well as the solutions of problems in which the plane the cross section varies within acceptable limits. This innovative approach builds on the substantial use of infinite points and the application of the *Swap of finite and infinite points* function and the refusal to use the affine property "the intersections of two parallel planes with a third plane are parallel lines". At the heart of this approach is the projective property of central projection to preserve the incidence of the main objects (points, lines and planes). Determine the piercing points of the lines on which lie all the edges of the parallelepiped with the plane of section α . It is easy to see that finding the intersections of the planes of two of the faces of a polyhedron with α is sufficient not only to determine the piercing points of the lines on which lie the edges of the parallelepiped incident to these planes, but also to ensure the existence of two common points of intersection of the planes of the other four faces of the parallelepiped with α . The remaining piercing points then can be determined.

Construction 1.3 For the convenience of the reader Construction 1.3 presented in (Tsareva & Zlatanov, 2016) will be described by the next task.

Problem 1. *Find the intersection of the parallelepiped $ABCD A' B' C' D'$ with the plane α , defined by the points P, Q and R , provided that two of the points P, Q, R lay on the plane of one and the same face of the parallelepiped $ABCD A' B' C' D'$ and the third point is arbitrary.*

The three, well known, theorems of Pap, Pascal and Brianchon used in the PhD thesis and are formulated.

Chapter I

System of tasks suitable for solving using DGS Sam

Chapter I discusses opportunities for research-oriented learning using DGS Sam. The advantages of studying geometric figures in a dynamic environment are well-established and indisputable. We offer a new approach to the study of topics from the school curricula, as well as a possible generalization of a task from the 53 International Mathematical Olympiad.

1.1 A Model of teaching through DGS in the study of the middle segment of a triangle

Interactive dynamic environments are among the new challenges for the teacher to motivate students for academic work. The main goal of innovation is to create a learning environment in which students can experiment with their ideas and fancies, to feel discoverers, reaching their own statements, to make generalizations, developing their abstract thinking, typical of mathematics.

Introduction of the concept of mean segment of a triangle, experimental establishment of its properties, formulation of statements and their proofs.

Problem 1. Sketch with the help of DGS Sam an arbitrary triangle ABC and its midpoint segment MN (M be a midpoint of AC , N be a midpoint of BC). Find connections between AB and MN .

The teacher comments on the possibilities for choosing the free objects. We comment on how to compare the lengths of AB and MN . The formulation of the hypothesis by the students themselves follows.

Theorem 1. A line passing through the midpoint of a segment of a triangle and parallel to the opposite segment intersects the third segment in its midpoint.

A proof is conducted with the active participation of the students. Then the teacher and the students prove the hypothesis and already formulate it as

Theorem 2. Every midpoint segment of a triangle is parallel and is equal to a half to a side of the triangle.

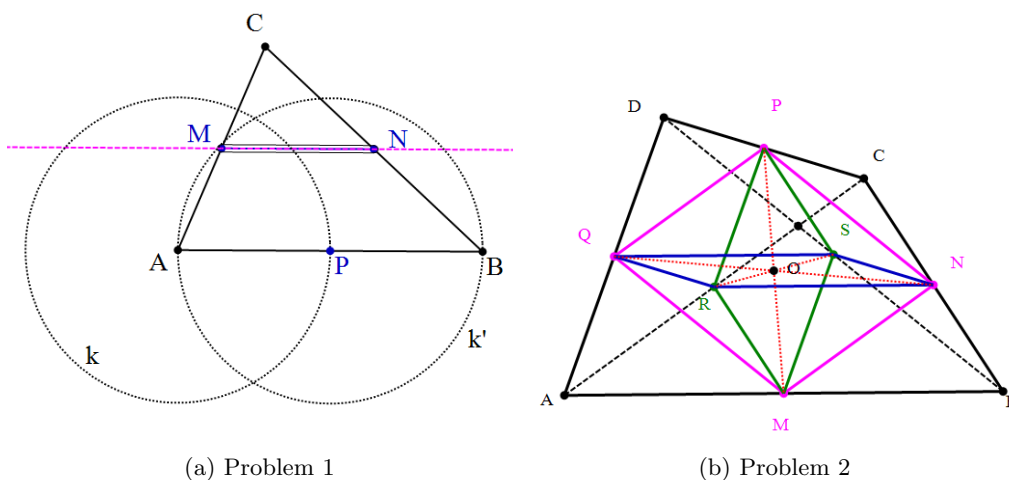


Figure 3

To exercise the acquired knowledge, several computational tasks with ascending gradation of difficulty are set.

There are 11 tasks in the textbook for 8th grade Lozanov et al. (2009), whose common root is the following task:

Problem 2. Let $ABCD$ be an arbitrary quadrangle. Let the points M, N, P, Q, R, S be midpoints of the segments AB, BC, CD, DA, AC, BD respectively. Prove that:

- the quadrangle $MNPQ$ is a parallelogram (Varignon's Theorem)
- the quadrangle $MSPR$ is a parallelogram
- the quadrangle $NSQR$ is a parallelogram
- the lines MP, NQ, RS are passing through a point.

The teacher introduces the notion of a midpoint segment of a quadrangle. The visualization assists the students in acquiring the new concept and in formulating their already proven property: *The three midpoint segments of the quadrangle intersect at a point.* The presence of infinite points in the DGS *Sam* and the *SFIP* function allows students to trace visually what happens to the parallelogram $NSQR$ and $MSPR$ when the quadrilateral $ABCD$ is transformed sequentially into a trapezoid (after swapping Q with Q_∞ , Figure 4a) and a parallelogram (after swapping P with P_∞ , Figure 4b).

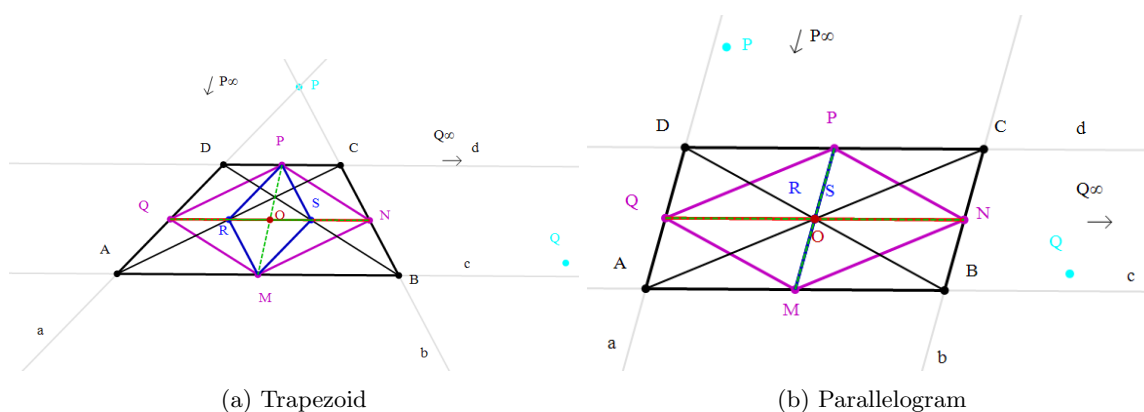


Figure 4. Problem 2

Problem 3. Let $ABCD$ be a quadrangle. Let the points M, N, P, Q be midpoints of the sides AB, BC, CD, DA , respectively. Prove that:

- $MNPQ$ is a rectangular if and only if $AC \perp BD$
- $MP = NQ$ if and only if $AC \perp BD$
- $MNPQ$ is a rhombus if and only if $AC = BD$
- $MNPQ$ is a square if and only if $AC \perp BD$ and $AC = BD$.

Students are prepared for the next tasks that are assigned for homework.

Problem 4. Sketch a dynamic drawing of a quadrangle $ABCD$, so that with the notations of Problem 3 the parallelogram $MNPQ$ to be: a) a rectangular; b) a rhombus; c) a square.

Problem 5. Let $ABCD$ be a quadrangle. Let the points M, N, P, Q be midpoints of the sides AB, BC, CD, DA , respectively. Prove that:

- $MSPR$ is a rectangular if and only if $AC \perp BD$
- $MSPR$ is a rhombus if and only if $AD = BC$
- $MSPR$ is a square if and only if $AD \perp BC$ and $AD = BC$.

Problem 6. Sketch a dynamic drawing of a quadrangle $ABCD$, so that with the notations of Problem 5 the parallelogram $MSPR$ to be: a) a rectangular; b) a rhombus; c) a square.

1.2 Application of DGS in the study of the centroid point of a triangle

Experimental detection of the centroid point properties can begin with the following task:

Problem 7. Sketch the medians of an arbitrary triangle.

The teacher assigns the following assignment:

Problem 8. Find out at what ratio the medians are divided by the centroid point.

It follows a formulation of the next theorem:

Theorem 3. The medians intersect at a point, which divides them in ratio $2 : 1$, considered from the vertex to the foot.

The active participation of students is stimulated again in proving this theorem. Applications of the acquired knowledge are presented in the next tasks.

Problem 9. Let $ABCD$ be a parallelogram. Let the points M, N, P and Q be midpoints of the segments BC, CD, DA and AB , respectively and O be the intersection point of the lines MP and NQ . prove that the points A, O and $X = BN \cap DM$ lie on a line (Figure 5).

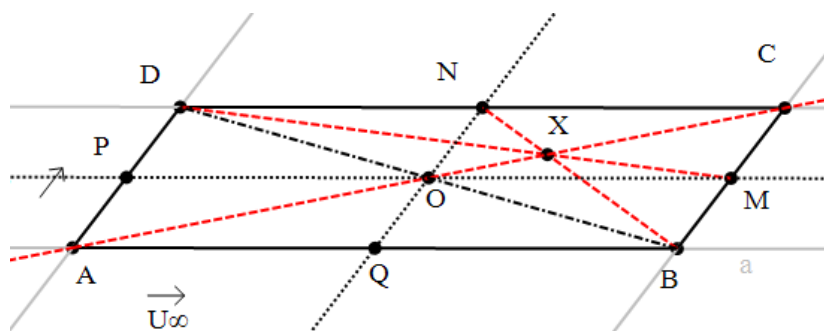


Figure 5. Problem 9

Problem 10. Let $ABCD$ be a parallelogram and O be a point in its interior. Let be constructed two lines passing through the point O being parallel to the sides of the parallelogram, let the intersection points of these lines with the sides of $ABCD$ be denoted with $M \in BC, N \in CD, P \in AD, Q \in AB$. Prove that the points A, O and $X = BN \cap DM$ lie on a line (Figure 6).

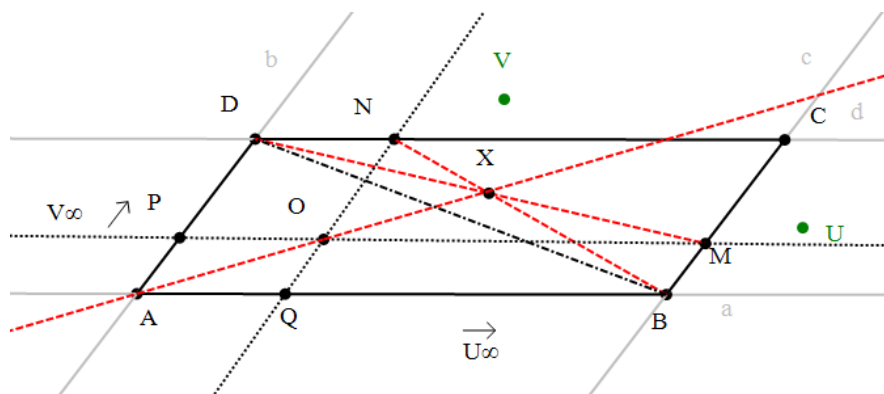


Figure 6. Problem 10

Students swap V_∞ with the arbitrary finite V and obtain trapezium, and then after swapping of U_∞ with the arbitrary finite U they get a quadrangle.

The dynamic sketches show that the points A , O and X continue to lie on one line. Together with the students, the problems concerning a trapezoid and an arbitrary quadrilateral are formulated.

Problem 11. Let $ABCD$ be a quadrilateral and O be a point in its interior. Let $AB \cap CD = U$, $BC \cap AD = V$. Let the lines OU and OV intersect the sides of $ABCD$ at the points $M \in BC$, $N \in CD$, $P \in AD$, $Q \in AB$. Prove that the points A , O and $X = BN \cap DM$ lie on a line.

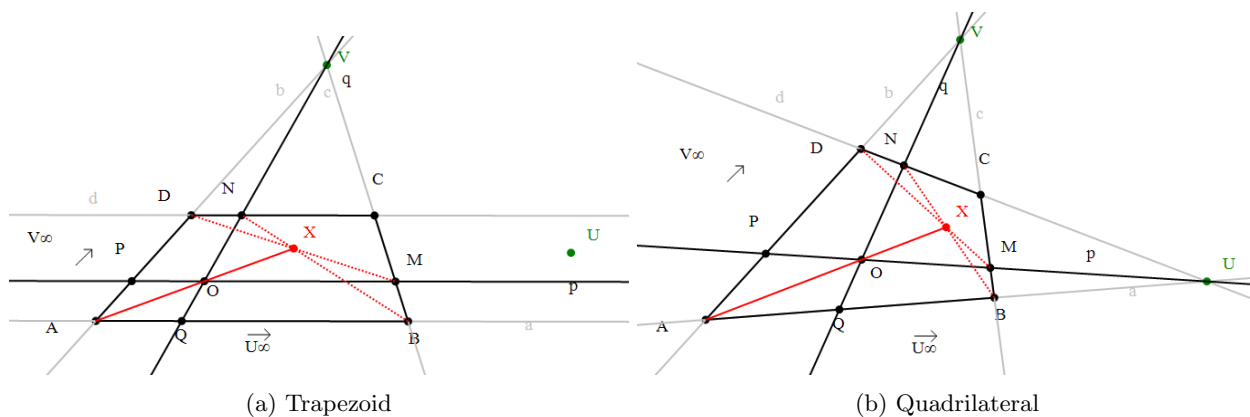


Figure 7. Problem 11

The last stage in the research follows – to prove the established facts. Students easily apply Pappus' Theorem to the two triplets (B, V, M) and (D, U, N) proposed by the teacher.

1.3 Development of creative abilities in students by studying the circumscribed quadrangles in a dynamic environment. Generalization of the task from the 53rd International Mathematical Olympiad, Argentina 2012

The present study shows how both the theoretical development of a topic and its presentation in the classroom can make teaching mathematics interesting, interactive, creative, independent or in the terms of (Knyazeva & Kurdyumov, 2010) "awakening".

Definition 1. We call a quadrangle circumscribed around a circle if all of its sides are tangent to the circle (Figure 8a).

Definition 2. A quadrangle such that the extensions of its sides are tangency to a circle is called exterior circumscribed around a circle (Figure 8b).

Problem 12. Sketch a quadrangle $ABCD$, circumscribed (exterior circumscribed) around a circle $k(I; r)$.

The construction of a quadrilateral circumscribed around a circle $k(I; r)$ can be reduced to the construction of the two pairs of tangents to $k(I; r)$ through two arbitrary external points X and Y for k . There are two options for choosing the points X and Y : Option 1 when X and Y are opposite vertices for the circumscribed (exterior circumscribed) quadrilateral and Option 2 when X and Y are the diagonal points P and Q of the circumscribed (Figure 8a) (exterior circumscribed (Figure 8b)) quadrilateral considered as a complete quadrilateral. Option 2 is preferred because it allows the use of the *SFIP* function.

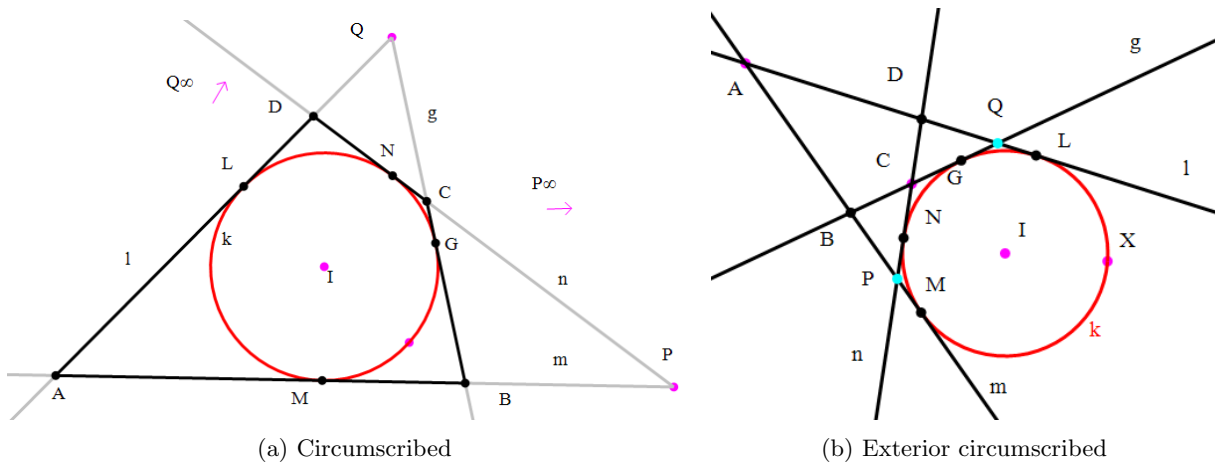


Figure 8. Circumscribed and exterior circumscribed

Let P_∞ and Q_∞ be two arbitrary infinite points. If we swap P with P_∞ , the sides AB and CD will become parallel and the circumscribed quadrangle $ABCD$ will be a trapezoid. After the second swap of Q with Q_∞ , the circumscribed trapezoid $ABCD$ is transformed into a parallelogram.

The pupils find that the bisectors pass through the centre I of the circle k , as for the circumscribed quadrangle l_a, l_b, l_c, l_d are bisectors of the inner angles, for the exterior circumscribed quadrangle l_a, l_c are bisectors at internal angles and l_b, l_d are bisectors at exterior angles.

Of course, the dynamic drawing only helps to identify and formulate the claims to be proven. Pupils themselves reach the first properties of a circumscribed and exterior circumscribed quadrangle.

Theorem 4. A quadrangle $ABCD$ is circumscribed around a circle if and only if the bisectors of three of its angles intersect at a point.

Theorem 5. A quadrangle $ABCD$ is exterior circumscribed around a circle if and only if the bisector of an interior angle and bisectors of its two neighbouring exterior angles intersect at a point.

Theorem 6. A necessary and sufficient condition for a quadrangle $ABCD$ to be circumscribed is the lengths of its sides to satisfy the equality $AB + CD = AD + BC$.

Theorem 7. A necessary and sufficient condition for a quadrangle $ABCD$ to be external circumscribed is the lengths of its sides to satisfy the equality $|AB - CD| = |AD - BC| \neq 0$.

Experimentally, students themselves come up with a statement concerning the concurrence of four special lines for the circumscribed (and exterior circumscribed) quadrilateral.

Theorem 8. *The diagonals of the circumscribed (external circumscribed) quadrangle and the lines connecting the tangency points of the opposite sides (the extensions of its opposite sides) with the circumscribed (external circumscribed) circle are intersecting at a point.*

The proof of this theorem by means of elementary mathematics is very extensive (4, 5 pages), while the double application of Brianchon's Theorem for the lines $mmlnng$ and $llmggn$ related to the quadrilateral $mlng$ solves the problem for both circumscribed (and exterior circumscribed) quadrangle.

The prominent students from 8 grade are already prepared to take part in a generalization of a problem from the 53-rd International Mathematics Olympics held in Mar del Plata, Argentina from 7 to 16 July 2012.

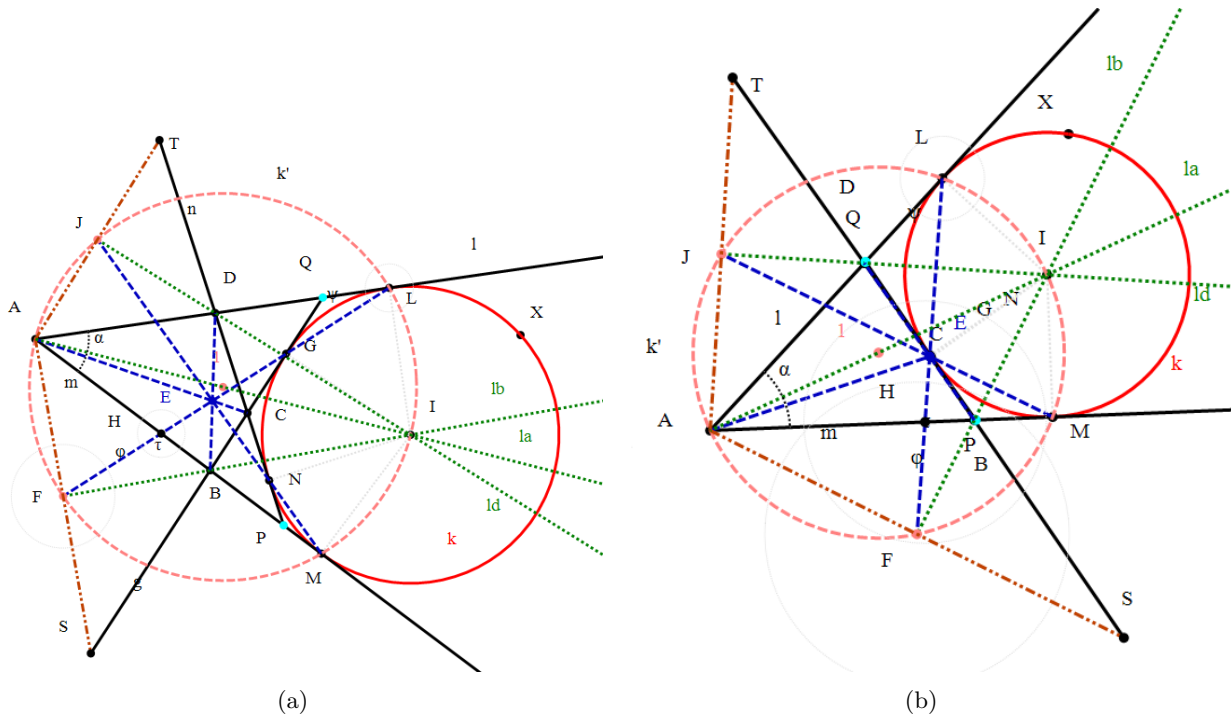


Figure 9. Problem 14

Problem 13. (*Evangelos Psychas, Greece*) *The centre of the exterior circumscribed circle tangent to the side BD of a given triangle $\triangle ABD$ is denoted by I . The circle tangencies to the lines AB, BD and AD at the points M, C and L , respectively. The lines LC and BI intersect at F and the lines MC and DI intersect at J . If S and T are the intersection points of BD with AF and AJ , respectively, then prove that C is the midpoint of the segment ST .*

The problem and its solution can be found in Magazine Mathematics +, 3, (2012), p. 51-52.

We will present a generalization of this Olympic problem for an external circumscribed quadrangle Tsareva & Todorova (2013).

Problem 14. *Let the quadrangle $ABCD$ be external circumscribed around a circle k with a centre I and let satisfies the conditions: $AB + BC = AD + DC$, $AB > CD$ and $AD > BC$. Let k*

be tangent to the lines AB , AD , BC and CD at the points M , L , G , N , respectively, the lines LG and BI intersect at the point F and the lines MN and DI intersect at the point J . Let S be the intersecting point of the lines AF and BC and T be the intersection point of the lines AJ and CD . Prove that the segments CS and CT have equal lengths.

Figure 9a, which is a stage from the dynamic sketch, in the creation of which the points M , N , L , $G \in k$ are chosen to be free. Problem 13 is a special case of Problem 14.

Figure 9a is a frame from the dynamic drawing, in the creation of which the points M , N , L , G and k are chosen to be free. Task 13 is a special case of Task 14.

1.4 Stimulation of creative activity in bilingual students through dynamic geometric software

This paragraph contains only the evaluation part of the publication Grozdev et al. (2014), in which I am a co-author.

Our experience confirmed the thesis that students with excellent mathematical training are convinced of the benefits of teaching mathematics with computers and have a positive attitude towards it. It is interesting to note that it is possible to create a positive attitude towards mathematics in students with negative attitudes towards the subject and low mathematical skills by learning mathematics with computers.

The results of Survey 1, before the training with the help of DGS Sam can be summarized as follows: Many students do not have a lasting knowledge of the elements of the quadrangles (vertices and sides), so they could not correctly determine the corresponding number; There is a lack of good knowledge of the names of the studied quadrangles. The notation of the names is not only full of spelling mistakes, but also suggests that their orthography has not been mastered. This is not an unexpected result for Roma students, who use their mother tongue predominantly in their speech practice.

We set ourselves the goal: Better mastery of the terms "parallel lines", "quadrangle", "trapezoid", "parallel", "rectangle", "rhombus", "square".

The results of the experiment show a significant change in the statistical results. We obtained that at an agreement level of 0.1 we can say that the distribution function changes so that the degree of mastery of the concepts of the type of quadrilaterals increases.

Chapter II

Research-oriented training with the use of DGS *GeoGebra*

Chapter II supports the thesis of the present work, but already using the dynamic geometric software *GeoGebra*. Possibilities for using *GeoGebra* in geometry classes, both to illustrate known tasks and to generate new ones, are discussed.

2.1 *GeoGebra* to develop students' creative thinking in geometry classes

The current research Zlatanov & Todorova (2019) is based on the following small school task:

Problem 15. Let ABC an acute triangle with an orthocenter H a centre I of its inscribed circle k . Find $\sphericalangle ACB$ if the points A , B , I , H lie on k . (Lozanov et al. (2009), Problem 15 b), p. 213))

It is essentially used in the solution of Problem ref prob6.1, that $\sphericalangle AHB = \sphericalangle AIB$, because by condition, that the points A , B , I , H belong to the circle k , i.e. the quadrilateral $ABIH$ is

inscribed. The question immediately arises: Is the point H obligatory to be an orthocenter? What will happen if we keep only the condition that the quadrilateral AB_1A_1B , where $A_1 = AH \cap BC$, $B_1 = BH \cap AC$, be inscribed. The check with a sketch made with *GeoGebra* immediately gave a positive answer and we formulated two new statements, generated by Task 15.

Let $\triangle ABC$ and the points $A_1 \in BC$, $B_1 \in AC$ be such that $\sphericalangle AA_1B = \sphericalangle AB_1B$ and $AA_1 \cap BB_1 = H$. Let AN be the bisector of $\sphericalangle BAC$, BM be the bisector of $\sphericalangle CBA$ and $AN \cap BM = I$. Let us introduce the notations: $\sphericalangle BAC = \alpha$, $\sphericalangle CBA = \beta$, $\sphericalangle ACB = \gamma$, $\sphericalangle AHB = \delta_1$, $\sphericalangle AIB = \delta_2$, $\sphericalangle AA_1B = \sphericalangle AB_1B = \varepsilon$.

Problem 16. *Let $\triangle ABC$ be arbitrary and let the points A_1 and B_1 satisfy the conditions A_1/BC and B_1/AC . Prove that the next conditions are equivalent:*

a) $\delta_1 = \delta_2$; b) $\varepsilon = \frac{3}{4}\gamma + \frac{\pi}{4}$; c) $\delta_1 = \frac{2}{3}\gamma + \frac{\pi}{3}$; d) $\delta_2 = \frac{2}{3}\gamma + \frac{\pi}{3}$.

let us consider the case when $\alpha \geq \frac{\pi}{2}$ and A/CB_2 , where B_2 is the intersection of the line AC and the circle k . Let us assume also that $\sphericalangle AHB_2 = \delta_3$.

Problem 17. *Let $\triangle ABC$ be an arbitrary and let the points A_1 and B_2 satisfy the conditions A_1/BC and A/CB_2 . Prove that the following conditions are equivalent:*

a) $\delta_2 + \delta_3 = \pi$; b) $\varepsilon = \frac{3}{4}\gamma + \frac{\pi}{4}$.

We would like to mention that sometimes when sketching tasks with *GeoGebra* it is necessary to construct them in a different way from the given conditions in the task. This gives additional knowledge to students and can be considered as an element of the topic of constructing geometric objects by given elements. A topic that is difficult for students, but with the use of DGS it creates an element of play and challenge, which stimulates the mastery of the curricula from the geometry course at school.

2.2 One generalization of the geometric problem from 19th Junior Balkan Mathematical Olympiad, Serbia 2015

Article citeStTo is another example of how strongly dynamic geometric software supports research work, because they allow to experimentally discover new relationships between the studied objects, followed by their theoretical justification. Interesting generalizations or discovery of new geometric objects related to the studied geometric configuration are often reached. In cases where the researcher's assumptions are not confirmed experimentally, a lot of unnecessary effort and time is saved.

We present a possible generalization inspired by the use of the DGS *GeoGebra* of the geometric problem from 19th Junior Balkan Mathematical Olympiad, June 24–29, 2015, Belgrade, Serbia.

The diagram for constructing formal proof is presented in Wong et al. (2011). We have generalized this diagram in the case when we search generalizations of task or new dependencies using DGS.

We introduced a group of students from ninth and tenth grade at the Mathematical High School "Academician Kiril Popov", Plovdiv, Bulgaria with the following geometric problem from the 19th Youth Balkan Mathematical Olympiad June 24–29, 2015, Belgrade, Serbia:

Following Wong et al. (2011), we drew the proof tree of Problem 18. We marked the key points of the proof in grey. Naturally, the question arose: The angle 90° , which is made by the lines l_1 and l_2 with the lines AB , as well as the lines h and g , respectively, with the lines AC and BC , essential is it for the statement $\sphericalangle ADB = \sphericalangle EMF$ or can it be arbitrary?

Problem 18. Let ABC be an acute triangle. The lines l_1 and l_2 are perpendicular to AB at the points A and B , respectively. Now let h and g be the perpendicular lines from the midpoint M of the side AB to the lines AC and BC . Let h and g intersect the lines l_1 and l_2 at the points E and F , i.e. $h \cap AC = E$ and $g \cap BC = F$. Let D be the intersection point of the lines EF and MC . Prove that $\sphericalangle ADB = \sphericalangle EMF$.

We formulated a hypothesis, which is a generalization of Problem 18.

Problem 19. Let ABC be an acute triangle and let the parallel lines l_1 and l_2 be going through the points A and B , respectively and be forming an angle α with the line AB . Also let M be the midpoint of the side AB . From point M we build the lines h and g , forming an angle α with AC and CB , respectively. Let us mark the intersecting points of the line h with the lines AC and l_1 with H and E , respectively, and the intersecting points of the line g with BC and l_2 with G and F , respectively. If D is the intersecting point of EF and MC prove that $\sphericalangle ADB = \sphericalangle EMF$.

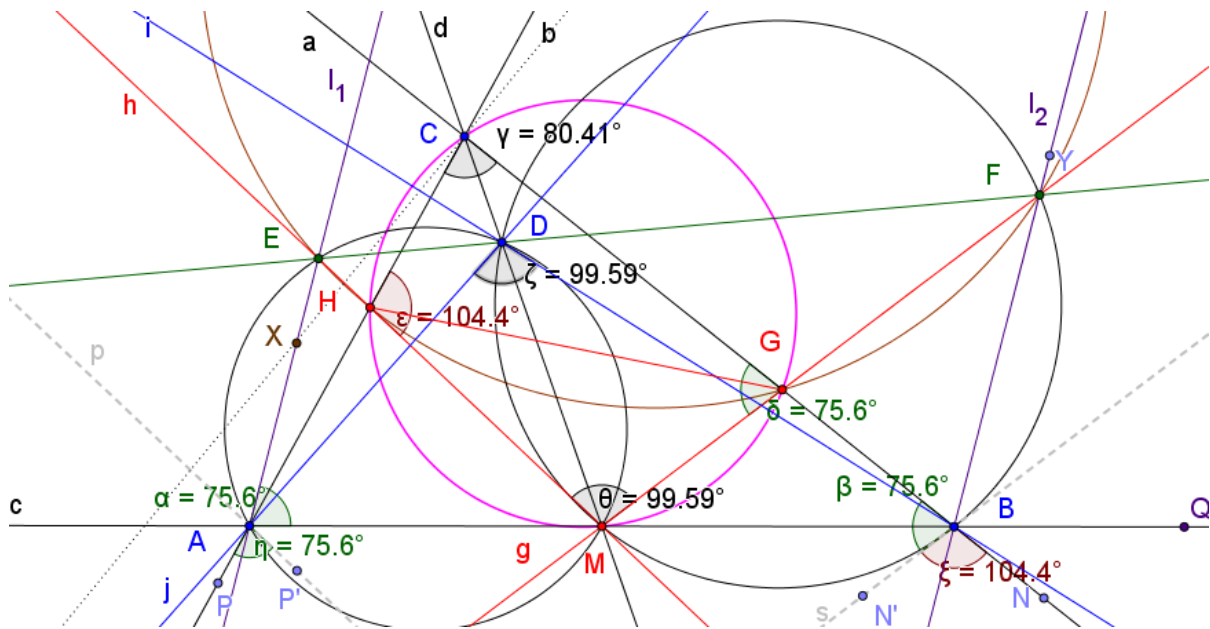


Figure 10

After the discussion to prove the generalized task, the students had the opportunity to write the tree of his proof.

We also considered the case when $H \neq ME$ and $G \neq MF$.

The following interesting fact also made an impression: At fixed $\triangle ABC$ the point D turned out to be fixed and is not affected by the change of angle α . We discussed this fact and clarifying it we came to some statements:

2.3 Intersections of polyhedrons with a plane

DGS can be used to illustrate intersections of planes with a plane, as well as to demonstrate the common root in solutions for seemingly differently defined intersection planes.

According to the curricula, students get acquainted with the method of intersections and the method of projections for finding the intersection of a polyhedron with a plane.

We used the technique proposed in Tsareva & Zlatanov (2016) to teach the topic "Intersection of a polyhedron with a plane" when studying it with students. An essential prerequisite for this application is the construction of the parallelepiped according to Construction 1.2, and the construction of the intersection according to Construction 1.3.

The plane of the intersection can be set in different ways, but they are all equivalent to setting three non-collinear points. There may be an infinite point between them, reflecting a condition for parallelism of certain objects. Let us assume that the plane of the intersection α is given by the points P, Q, R . Without loss of generality, we can assume that the points P and Q lay in the plane of the faces $ABCD$.

2.3.1 The method of intersection planes

We start with the simplest case of finding a plane section of a polyhedron, when the plane of the section is given by three points lying on straight lines, incident to three edges with a common vertex, to introduce students to some additional possibilities provided by *GeoGebra* for a clearer view of the plane sections of a polyhedron.

Problem 20. Find the intersection of the parallelepiped $ABCD A' B' C' D'$ with the plane $\alpha(P, Q, R)$, provided that $P \in BC, Q \in AB, R \in BB'$.

Using the dynamics of *GeoGebra*, we notice that when moving the points P, Q and R , respectively on the segments BC, AB and AA' the intersection continues to be the triangle PQR , regardless of the position of the points P, Q and R , respectively on the edges BC, AB and BB' .

Now again only with the method of intersection planes we will solve a more general problem when the points defining the plane of the intersection α are arbitrary points lying in two of the walls of the universal parallelepiped.

Problem 21. Find the intersection of the parallelepiped $ABCD A' B' C' D'$ with the plane $\alpha(P, Q, R)$, provided that P and Q are arbitrary points on the plane of the wall $\gamma(ABCD)$ and R be an arbitrary point on the plane of the wall $\delta(ADD' A')$.

Find the intersection of the plane $\alpha(P, Q, R)$ with all of the connected figures of the parallelepiped $ABCD A' B' C' D'$.

The solution of the first part of the problem is realized immediately as a result of the application of Construction 1.3

To solve the second part of the problem, we use the applet *SFIP*.

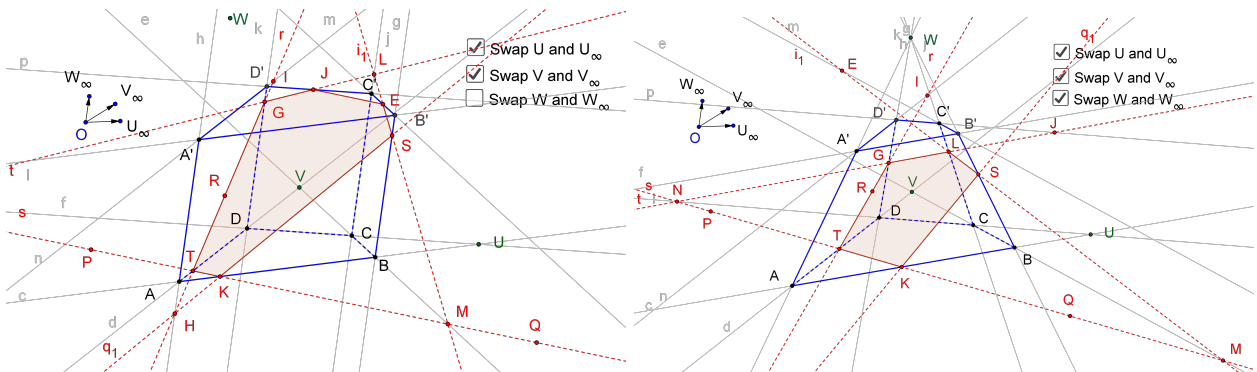


Figure 11. $ABCD A' B' C' D'$ be a prism with a base of an arbitrary quadrilateral or a truncated pyramid with a base of an arbitrary quadrilateral

With a little more efforts it is possible to illustrate the problem when the point Q is replaced by the infinite point of the line BC , which we denote by Q_∞ . Figure 11 shows the sections of the plane $\alpha(P, Q_\infty, R)$ with two connected figures of the parallelepiped $ABCD A' B' C' D'$.

2.3.2 The projection method makes

The projection method makes it more difficult for students than the intersection planes method.

Problem 22. Find the intersection of the parallelepiped $ABCD A' B' C' D'$ with the plane $\alpha(P, Q, R)$, provided that two of the points P and Q lay on the plane of one and the same face of the parallelepiped $ABCD$ and the third one R be an arbitrary point.

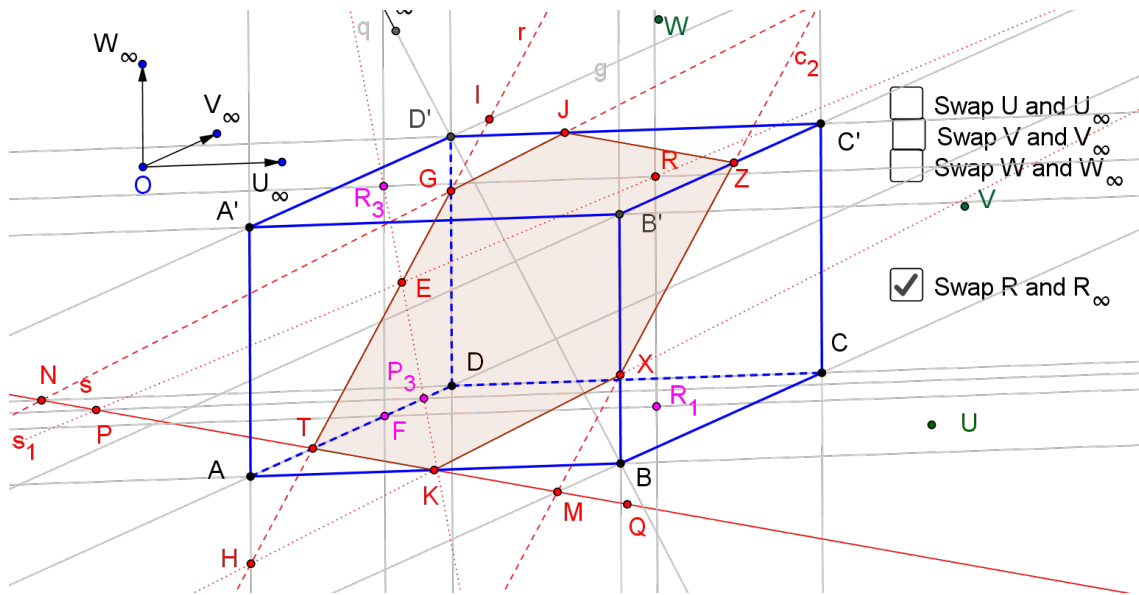


Figure 12. Problem 22

We are looking for a second point from the intersection line r of the planes $\alpha(P, Q, R)$ and $\delta(ADD'A')$. There is already a known point of this intersection line, namely T . The projections of the points P and R on the plane δ along U_∞ are P_3 and R_3 , respectively. Then the point $E = PR \cap P_3R_3$ is the sought second point from the intersection line $r = \alpha \cap \delta$ (Figure ref fig: sec1-13).

If $P \in AD$, then $P = P_3 = T = E$ and the line TE cannot be constructed. The points P and Q are equal, so we can construct the projection $Q_3 = QU_\infty \cap AD$. Then $E' = QR \cap Q_3R_3$ is the required second point of the intersection line $r = \alpha \cap \delta$.

The following is a series of tasks that specify the location of some of the points P , Q and R .

The innovative method proposed by Problem 22 allows a rapid construction of the plane intersection with the universal parallelepiped and all of its connected figures.

Problem 23. Find the intersection of the parallelepiped $ABCD A' B' C' D'$ with the plane $\alpha(P, Q, R)$, provided that Q be the midpoint of the segment AB , P be in the plane of the face $\gamma(ABCD)$ and R be the intersection point of the diagonals of the parallelepiped, i.e. $R = AC' \cap BD'$.

We redefine $R = AC' \cap BD'$ and $Q = \text{Midpoint}(c)$. The intersection is obtained automatically (Figure ??). We notice that the line TE is parallel to QR . We prove with the students that this is indeed the case with an arbitrary choice of the point $P \in \gamma(ABCD)$.

When the condition P is a midpoint of BC is added to the wording of Problem 23, the solution is obtained automatically.

Getting acquainted with the innovative approach based on Construction 1.3 does not mean abandoning the classical methods included in the curriculum, but only their enrichment. In support of this thesis, consider the following task

Problem 24. *Let $ABCD A' B' C' D'$ be a truncated quadrilateral pyramid. Find its intersection with the plane α , which is incident with the midpoints of the edges AB and BC and is parallel with the line BB_1 , where B_1 is an arbitrary point from the edge DD' .*

Two solutions of Problem 24 are presented. The first is the classic, which is used in school education, and the second is based on the basic construction of Problem 22. We comment with the students that the two constructions give one and the same intersection.

2.3.2 An improvisation on Problem 22, when the parallelepiped is replaced with a pyramid

We apply Problem 22 in solving tasks for finding intersections of a plane with a pyramid or a double pyramid with an additional requirement for the type of the intersection. We consider the cases when the intersection is a parallelogram, a rectangle, a rhombus or a square. The results of this paragraph are published in (Teofilova & Todorova, 2020).

Problem 25. (Main Problem) *Select the plane α , such that its intersection $HGLS$ with the pyramid $ABCDW$, where $ABCD$ is an arbitrary quadrangle, is a parallelogram.*

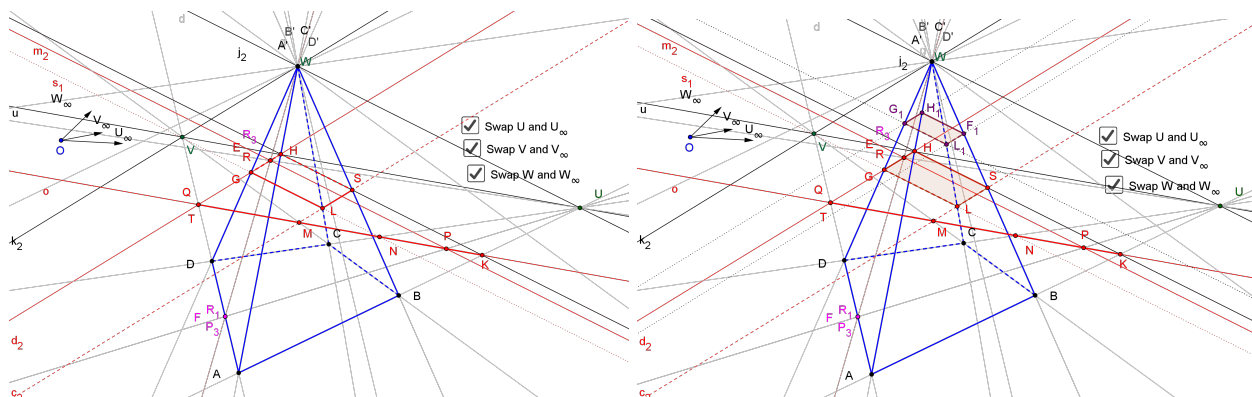


Figure 13. Problem 25 and Problem 26

Problem 26. *Construct the intersection $H_1 G_1 L_1 S_1$ with the pyramid $ABCDW$, which is parallel to the intersection $HGLS$.*

Problem 27. *Select the apex W of the pyramid $ABCDW$, where $ABCD$ is an arbitrary quadrangle so that its intersection $HSLG$ with the plane α be a rectangle.*

Problem 28. *Select the apex W of the double pyramid $ABCDW A' B' C' D'$, where $ABCD$ is an arbitrary quadrangle, such that its intersection $HSLG$ of the plane α with the upper part of the double pyramid to be a rhombus.*

Problem 29. *Select the apex W of the pyramid $ABCDW$, where $ABCD$ is an arbitrary quadrangle, such that its intersection $HGLS$ with the plane α be a square.*

Using their creative potential, students can formulate and obtain solutions to a series of new problems, including truncated pyramids, as well as more than two specialized parallel sections.

Chapter III

Diagnostic procedure for control of students' knowledge and skills

3.1 Theoretical statement

As mentioned in the previous chapter, the topic was chosen for „*Intersection of a polyhedron with a plane*“ to convey it using a dynamic geometric environment and then to conduct a diagnostic procedure in order to evaluate the test we created, but also to confirm the hypothesis made by us at the beginning of the dissertation.

Years of experience as a teacher has convinced me of the usefulness of the situation in which the student becomes an active participant in the learning process. The use of DGS allows students to feel like discoverers when the chosen topic is favourable for the application of DGS. The current diagnostic procedure addresses the problem of organization and implementation of system control by the teacher when studying the topic "Intersection of a polyhedron with a plane" in a dynamic geometric environment. The examination of students' knowledge is a learning activity, specially organized by the teacher, through which the results of the learning process are established. Diagnosis is an ongoing process through which, on the one hand, the mathematical development of students is established, and on the other hand, regularities are deduced and changes are identified to improve the teaching methodology.

3.1.1 Analysis of the curriculum in mathematics in the 11 form (second level)

The present paper describes a diagnostic procedure in the study of figures and bodies, and in particular Theme 5. Stereometry. Polyhedra, with the help of modern software products. Therefore, we will pay special attention to the standards and expected results at curriculum level on this topic.

When studying the topic of cross section of a polyhedron with a plane, students must skilfully use the basic definitions, axioms and theorems of stereometry.

In preparing the topic, we have adhered to (Dodunekov et al., 2002) that is used in the school classes.

3.1.2 Verification and assessment of knowledge through tests

The problem of testing and assessing students' knowledge and skills has never lost its relevance. New forms and means are being sought to verify the acquired knowledge. The assessment must be objective and justified.

Didactic tests are suitable as a method for checking and assessing the students' achievements in studying a certain educational content. The didactic test is subject to strictly defined requirements for objectivity, reliability, content and validity. Special procedures have been developed to verify these qualities.

The didactic test is a standardized system of questions and tasks, checking in equal conditions the level of training or readiness for training. It is a kind of achievement test (Ivanov, 2006)

The wide penetration of didactic tests in the learning process is determined by the qualities they possess: objectivity, validity, reliability, difficulty and content. Combined with other forms and methods of control, the tests contribute to the active participation of all students in the learning process on equal terms. In addition to provoking the activity, they arouse the curiosity and cognitive interests of students, shorten the time for checking, give an accurate idea of the degree of assimilation of knowledge by students. Didactic tests must correspond in content and form to the age and psycho-physiological characteristics of the tested students.

Non-standardized tests are mainly used in diagnostic tests. They are created by teachers for the needs of education. Particular attention is paid to defining the goals and requirements for learning specific content. When constructing the test, attention is paid to the selection of different types and structure of tasks, their combination and arrangement.

3.2 Research program

The purpose of the diagnostic procedure is: Development and research of a didactic test for diagnosing the knowledge and skills of students in studying the theme "Intersection of a polyhedron with a plane" from the curriculum in mathematics in 11th grade (profiled training) in a dynamic geometric environment.

The object-subject area of the diagnostic procedure is directly dependent on that of the PhD thesis and the achievements of students (knowledge, skills, competencies) through the use of dynamic geometric software to increase the level of competence and build skills for its application in finding planar sections of polyhedra in the classes of Mathematics – 11th grade (profiled training). The subject of the diagnostic procedure is a study of the qualities of a diagnostic test conducted in the DGS to check the knowledge and skills of students on the topic "Intersection of a polyhedron with a plane" in the classes of Mathematics - 11th grade (profiled training).

The research was conducted with students from four classes from 11 form of Mathematical High School "Academician Kiril Popov", Plovdiv, studying in classes with a natural sciences and mathematics profile and intensive study of English or German or French. The classes are composed respectively: 27 students, 28 students, 26 students, 25 students. Five of the students do not participate in the testing due to health problems.

The main methods included in the diagnostic test are: *tests, surveys, expert evaluation, statistical methods for processing the results.*

The criteria for evaluation of the research must meet certain requirements: completeness, unambiguity, constructiveness, to reflect the essence of the research process, not to be covered by content.

Criteria formulate a general situation, namely the correspondence between planning and achieved results. Then we go to decomposition to results, acquisition of knowledge, mastering skills. Indicators are also decomposed to the extent of this knowledge, the degree of meaningfulness and mastery. Finally, each indicator is expressed numerically.

3.3 Presentation and analysis of the results of the diagnostic procedure

To perform the diagnostic test, we apply the methods of testing and questionnaires.

The activities for carrying out didactic tests are related to the preparation of a test, to the study of its characteristics - reliability, validity, difficulty and content. A test within 16 tasks has been created for the purposes of this diagnostic procedure. The characteristics are studied and an expert assessment is made.

Purpose of the test: To diagnose the acquired knowledge and skills of students in studying the topic "Intersection of a polyhedron with a plane" and to identify their gaps and difficulties.

The two main methods for constructing a section of a prism and a pyramid with a plane are covered, namely the method of intersections and the method of projections. Triangular and quadrangular prism and pyramid tasks are included, as well as one truncated pyramid task to cover the entire volume of the polyhedra known to students so far. There are problems in which the plane is determined by 3 points, as well as by a point and a line. In the choice of the three points there are problems in which the two points lay in one and the same face of the polyhedron and those in

which no two points to be in one and the same face of the polyhedron. Tasks are selected in which at least one of the points does not lay on a line incident with an edge of a polyhedron.

Students receive a current grade from the test result. Since its main goal is for the teacher to diagnose the teaching of the theme using the dynamic geometric software *GeoGebra*, students are encouraged to show maximum knowledge on the topic, each of the tasks is evaluated with an equal number of points - 1 point.

Due to the Covid-19 pandemic, the test was conducted online with *Google Forms*.

A Study of the characteristics of the test

Specification of the test tasks - an assessment is made to what extent the tasks correspond to the learning objectives.

According to Bloom's taxonomy (Bloom et al., 1956), created in 1956, learning cognitive goals are divided into six main groups – knowledge, understanding, application, analysis, synthesis and evaluation. For the purposes of this study, some of the categories are combined and three categories of mental activity are adopted: knowledge, understanding, and application. Based on the formulated goals of the specific educational content, a table was created.

Psychometric characteristics of the test tasks - the test is performed. The total score of each student is calculated as a sum of the received points and the students are arranged in ascending order. Then the students are divided into two groups - weak and strong, with the same number of members. Calculate:

✓ *Task difficulty index* – the difficulty of the tasks is measured by the extent to which and whether they are solved by the students. To determine the difficulty index, the formula $P = \frac{N_R}{N} \cdot 100$ is used, where N_R is the number of students who solved the problem correctly and N is the total number of all tested who solved the problem.

✓ *Coefficient of discriminant force of the task* – shows the extent to which the tasks distinguish the strong group in which students demonstrate more stable knowledge from the weak group for students in which each task is more difficult. The coefficient of discriminant force of the task is determined by the formula: $DP = \frac{R_U - R_L}{\frac{1}{2}T}$, where R_U is the number of students from the "strong" group who have solved the task correctly, R_L is the number of students from the "weak" group who have solved the task correctly and T is the total number of students in both groups. The average coefficient of the test is 0.45.

✓ *Distractor analysis* Based on the results of the test, information was collected on how many times each of the distractors was indicated, both by the students in the weak group and by those who fall into the strong one.

From the data it is clear that in almost all questions, each of the distractors was chosen more times by students from the weak than in the strong group.

✓ *Test validity* - this is an indicator of the degree of correspondence between what the test measures and the objectives of the measurement itself. In the present development the content and criterion validity of the test is determined.

✓ *Test reliability* - this is one of the most important features of the test. The higher the reliability, the smaller the measurement error made with the corresponding test. The method of parallel forms was applied to analyse the reliability of the test. To avoid re-testing, we divided the tasks into even and odd numbers, evaluating and comparing them. This way of conducting a criterion test is used in (Staribratov, 2012). The reliability of a test shows how accurate it is to measure what we set out to measure. The value of the reliability coefficient is high, which indicates that the test is generally reliable.

Processing of the results

The test is not a universal diagnostic tool, so its results should not be accepted unreservedly and uncritically. There are different approaches to evaluating test results.

To determine the level of acquired knowledge and master the skills, the test tasks are analysed.

Each test is processed separately, and the data obtained is presented in a table called the primary protocol (Appendix №4), in which the numbers of students and the number of points obtained for each task are entered. For each student the total test score is calculated - the sum of the points obtained for each task and an individual coefficient according to the formula: $K_{ind} = \frac{a}{b}$,

where a – test score of the specific student, b – total test score, $K_{cp} = \frac{\sum K_{ind}}{n}$.

We arrange the obtained statistical order of test scores and individual coefficients in ascending order and group the repeating values. In this way we get a frequency distribution.

The main characteristics of the frequency distribution are the indicators of the central trend and the scattering.

Indicators of the central trend are the arithmetic mean, median and mode.

Arithmetic mean (\bar{X}) – the sum of the total test scores of all students studied is divided by the sample size.

Median (M_e) – the point that bisects the statistical order. After arranging the statistical order in ascending order is this test score, which is in the middle and halves it. If the sample size is an even number, the median is the half-sum of the two average test scores.

Mode (M_o) - the most common variant in the ranking order. Ideally, the indicators of the central trend coincide.

The test results are good. They are between 41% and 88%. Poor grades are kept to a minimum. The training on the topic is carried out with the help of dynamic geometric software, which is a prerequisite for good results.

During the years of experience, the poor grades (fail 2 or satisfactory 3) on the theme of "Intersection of a polyhedron with a plane" have always been about 35%. The current results are significantly higher, which shows that the use of DGS in teaching and assessment on the topic increases the success of students with less interest in mathematics.

The increase in results may be due to two factors:

- well-constructed didactic test
- the use of dynamic geometric software.

With the help of Information Technology, the interest of students with low results in mathematics and with a negative attitude towards the subject increases.

The analysis of the results of the conducted diagnostic procedure shows that the applied methodology gives good results in mastering the curriculum and assessing the knowledge and skills of students on the theme "Intersection of a polyhedron with a plane" with the help of DGS.

From all that has been said so far, it can be argued that the formulated hypothesis is correct.

Conclusion

Summary of the obtained results

The current generation of students is not only prepared to actively use computers in their education, but they are eager to apply their digital skills in the learning process (Zlatanov, 2018). It

is an indisputable fact that the mathematical knowledge obtained through independent observation and research is deeper and more durable.

The presence of heuristic moments and the use of dynamic geometric software makes learning fascinating, attractive, modern and forms a positive attitude towards mathematics. Following this approach, the teacher teaches not only knowledge but also methods of research thinking, it stimulates students' creativity and enriches their mathematical culture.

We would like to say that the use of DGS in geometry classes brought a new mood and spirit to them. Students begin to work more carefully. Increases both interest and results. It allows for optimization of the drawing work and the acquisition of a larger amount of knowledge. With the illustrated examples we consider that the use of The DGS is suitable both for illustrating the theorems and problems from the school material and for stimulating the creative activity of the students by self-discovery and formulation of known results and by summarizing such, both standard tasks from textbooks and of tasks with increased difficulty from competitions. Using the function of exchanging endpoints and infinite points makes it possible to find common roots in the evidence of problems that at first glance are different.

The opportunity to experimentally self-discover the connections between the studied objects encourages students to continue researching, and the more talented to create. Dynamic geometric software allows a creative learning process and deeper understanding of geometry. It is especially useful for studying and visualizing 3D geometric objects and their properties. The use of DGS also creates an element of play and challenge, which stimulates the mastery of the material from the geometry course at school. To the widely recognized advantages of learning using dynamic geometric software we must add the positive psychological effect, evidenced by:

- the joy of the students from the feeling of freedom, characteristic of the young people
- the joy of the students from their empathy in the discovery.

A clear sign of this is their desire for maths classes to continue in the same format.

The positive results encourage us to increase the volume of topics in the geometry curriculum that we will teach students by applying dynamic geometric software. Then the analyses will give a more accurate assessment of the benefits of studying geometry in a dynamic environment.

Contributions of the PhD thesis

The main contributions, according to the author, in the present PhD thesis are:

- I. Developed systems of tasks and methodological approaches for the use of DGS in geometry classes of 5th, 8th, 9th and 11th forms
- II. Application of DGS for research-oriented teaching in geometry to students – bilinguals and students with increased interest in mathematics
- III. Creating a didactic environment and learning technology through the application of DGS for the formation of creative thinking of students in geometry classes in 11th form
- IV. Development of methodological tools for training on the theme „Intersection of a polyhedron with a plane“
- V. Development of a system of tasks using DGS for training on the theme „Intersection of a polyhedron with a plane“, when specializing the type of section

VI. Test form has been created for performing a diagnostic procedure with subsequent full analysis of the results.

Prospects for future research development:

- Increase the topics in which the DGS will be used to illustrate and search for new summaries
- Development of didactic tests for other topics in the school curriculum
- Involvement in mathematics training to illustrate and support heavy computations of the algebraic computer system Maple, in parallel with the DGS
- Sharing good practices for using computer systems with fellow math and computer science teachers.

List of publications included in the PhD thesis

- 1 Petya Trifonova, Nikolina Danova, **Radka Zlatanova**, Research-oriented teaching on the topic "The mid-segment and the centroid of a triangle" in a dynamic geometry environment, *The Educational Forum* **1**(4), 94–105, 2013, ISSN:1314-7986 (ERIH PLUS) (in Bulgarian).
- 2 Bistra Tsareva, **Radka Todorova**, Interactive study of circumscribed quadrilaterals in dynamic environment, *Mathematics and Informatics*, **56**(2), 142–158, 2013, ISSN: 1310-2230, eISSN1314-8532 (Web of Science, Accession Number WOS:000447344600005) (in Bulgarian).
- 3 Sava Grozdev, Diana Stefanova, Kalina Vasileva, Stanislava Koleva, **Radka Todorova**, Creative activity stimulation of bilingual students by dynamic software, *Mathematics and Informatics*, **75**(3), 247–273, 2014, ISSN: 1310-2230, eISSN1314-8532 (Web of Science, Accession Number WOS:000450489800002) (in Bulgarian).
- 4 Boyan Zlatanov, **Radka Todorova**, Using GeoGebra for the development of creative thinking of Students in geometric classes, *Information technologies in education - challenges and opportunities: collection of scientific reports from the Scientific-practical conference "Mathematics, informatics, information technologies, application in education"*, 228–235, 10–12 October, Pamporovo, ISBN:978-619-202-437-6 (in Bulgarian).
- 5 Ivailo Starilratov, **Radka Todorova**, One generalization of the geometric problem from 19th Junior Balkan Mathematical Olympiad, *Mathematics and Informatics*, **62**(2), 204–216, 2019, ISSN: 1310-2230, (Web of Science, Accession Number WOS:000465591000006).
- 6 Marta Teofilova, **Radka Todorova**, A New Improvisation on the Topic „Intersection of a Pyramid with a Plane“, *Global Journal of Advanced Research on Classical and Modern Geometries*, **9**(1), 15–19, 2020 (MathSciNet).

Approbation of the obtained results

Science Conference: Information technologies in education - challenges and opportunities: collection of scientific reports from the Scientific-practical conference "Mathematics, informatics, information technologies, application in education", Pamporovo, 7–8.11.2019.

The connection between the contributions, the tasks, the paragraphs in the PhD thesis and the included publications.

The connection between the contributions, the tasks, the paragraphs in the PhD thesis and the included publications is as follows:

Contribution	Paragraph or Chapter	Publication	Conference report
I	2.1,2.2,2.3,2.4,3.1,3.2,3.3	1,2,3,4,5,6	1
II	2.1,2.2,2.3,3.1,3.2,3.3	1,2,4,5,6	1
III	2.3,3.1,3.2	2,4,5	
IV	3.3	6	
V	3.3	6	
VI	2.4	3	

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